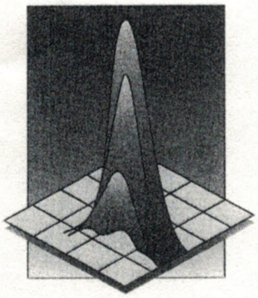
138



SP Forum

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History and Equivalence of Two Methods of Spectral Analysis

The purpose of this article is to present a brief history of two methods of spectral analysis and to present, in a tutorial fashion, the derivation of the deterministic relationship that exists between these two methods.

History

Two of the oldest and currently most popular methods of measuring statistical (average) power spectral densities (PSD's) are the frequency smoothing method (FSM) and the time averaging method (TAM). The FSM was thought to have originated in 1930 with Norbert Wiener's work on generalized harmonic analysis [1], and to have been rediscovered in 1946 by Percy John Daniell [2]. But, it was discovered only a few years ago (cf. [3]) that Albert Einstein had introduced the method in 1914 [4]. The currently popular method of deriving the FSM begins by showing that adjacent frequency bins in the periodogram have approximately the same correct mean values and the same large variances, and are approximately uncorrelated with each other. Then, it is observed that averaging these bins together retains the correct mean value, while reducing the variance.

The TAM is often attributed to a 1967 paper by P.D. Welch in the *IEEE Transactions on Audio and Electroacoustics* [5], but in fact the earliest known proposal of the TAM was by Maurice Stevenson Bartlett in 1948 [6]. The reasoning behind the TAM is similar to that for the FSM: the periodograms on adjacent segments of a data record have approximately the same correct mean values and the same large variances, and they are approximately uncorrelated with each other. Therefore, averaging them together will retain the correct mean value, while reducing the variance. (A more detailed historical account of the FSM, TAM, and other methods is given in [7].)

Essentially, every spectral analysis software package available today includes either the FSM or the TAM, or both, often in addition to others. These other methods include, for example, the Fourier transformed tapered autocorrelation method, attributed to Ralph Beebe Blackman and John Wilder Tukey [8] (but used as early as 1898 by Albert A. Michelson [9]); and various model fitting methods that grew out of pioneering work by George Udny Yule in 1927 [10] and Gilbert Walker in 1931 [11].

It is well known that both the FSM and the TAM yield PSD estimates that can be made to converge to the exact PSD in some probabilistic sense, like in mean square as the length of the data record processed approaches infinity, However, it is much less commonly known that these two methods are much more directly related to each other. The pioneering methods due to Michelson, Einstein, Wiener, Yule, and Walker were all introduced without knowledge of the concept of a stochastic process. But starting in the 1950s (based on the work of mathematicians such as Khinchin, Wold, Kolmogorov, and Cramér in the 1930s and 1940s), the stochastic-process point of view essentially took over. It appears as though this mathematical formalism, in which analysts focus on calculating means and variances and other probabilistic measures of performance, delayed the discovery of the deterministic relationship between the FSM and TAM for about 40 years. That is, apparently it was not until the non-stochastic approach to understanding statistical (averaged) spectral analysis was revived and more fully developed in [7] that a deterministic relationship between these two fundamental methods was derived.

The next section presents, in a tutorial fashion, the derivation of the deterministic relationship between the FSM and TAM, but generalized from frequency-smoothed and time-averaged versions of the periodogram to same for the biperiodogram (also called the cyclic periodogram [7]). This deterministic relationship is actually an approximation of the time-averaged biperiodogram (TAB) by the frequency-smoothed biperiodogram (FSB) and, of course, vice-versa. For evidence of the limited extent to which this deterministic relationship is known, the reader is referred to letters that have appeared in the *SP Forum* section of this magazine in the October 1994, January 1995, March 1995, and November 1995 issues.

Equivalence

Definitions

Let $a(t)$ be a data-tapering window satisfying $a(t) = 0$ for $|t| > T/2$, let $r_a(\tau)$ be its autocorrelation

$$r_a(\tau) = \int_{-T/2}^{T/2} a(t+\tau/2)a(t-\tau/2)dt$$

and let $A(f)$ be its Fourier transform

$$A(f) = \int_{-T/2}^{T/2} a(t) e^{-i2\pi f t} dt$$

Let $X_a(t, f)$ be the sliding (in time t) complex spectrum of data $x(t)$ seen through window a,

$$X_a(t, f) = \int_{-T/2}^{T/2} a(w) x(t+w) e^{-i2\pi f(t+w)} dw$$

Similarly, let $b(t)$ be a rectangular window of width V, centered at the origin, and let $X_b(t, f)$ be the corresponding sliding complex spectrum (without tapering). Also, let $R_a^\alpha(t,\tau)$ be the sliding cyclic correlogram for the tapered data

$$R_a^\alpha(t,\tau) = \int_{-(T-|\tau|)/2}^{(T-|\tau|)/2} a(v+\tau/2) x(t+[v+\tau/2]) \cdot a(v-\tau/2) x(t+[v-\tau/2]) e^{-i2\pi\alpha(t+v)} dv$$

and let $R_b^\alpha(t,\tau)$ be the sliding cyclic correlogram without tapering

$$R_b^\alpha(t,\tau) = \frac{1}{V} \cdot \int_{-(V-|\tau|)/2}^{(V-|\tau|)/2} x(t+[v+\tau/2]) x(t+[v-\tau/2]) \cdot e^{-i2\pi\alpha(t+v)} dv$$

To complete the definitions, let $S_a(t; f_1, f_2)$ and $S_b(t; f_1, f_2)$ be the sliding biperiodograms (or cyclic periodograms) for the data $x(t)$,

$$S_a(t; f_1, f_2) = \frac{1}{T} X_a(t, f_1) X_a^*(t, f_2)$$

$$S_b(t; f_1, f_2) = \frac{1}{V} X_b(t, f_1) X_b^*(t, f_2)$$

Derivation

It can be shown (using $\alpha = f_1 - f_2$) that (cf. [7, Chapter 11])

$$\begin{aligned}
\frac{1}{V} &= \int_{-V/2}^{V/2} S_a(t-u; f_1, f_2) du \\
&= \frac{1}{V} \int_{-V/2}^{V/2} \int_{-T}^{T} R_a^\alpha(t-u,\tau) du\, e^{-i\pi(f_1+f_2)\tau} d\tau du \\
&= \int_{-T}^{T} \frac{1}{V} \int_{-V/2}^{V/2} R_a^\alpha(t-u,\tau) du\, e^{-i\pi(f_1+f_2)\tau} d\tau \\
&\cong \int_{-T}^{T} R_b^\alpha(t,\tau) r_a(\tau) e^{-i\pi(f_1+f_2)\tau} d\tau \\
&= \int_{-\infty}^{\infty} S_a(t; f_1 - g, f_2 - g) \left[\frac{1}{T} |A(g)|^2\right] dg
\end{aligned}$$

The above approximation, namely

$$\frac{1}{V} \int_{-V/2}^{V/2} R_a^\alpha(t-u,\tau) du \cong R_b^\alpha(t,\tau) r_a(\tau)$$

for $|\tau| \leqslant T$, becomes more accurate as the inequality $V >> T$ grows in strength (assuming that there are no outliers in the data near the edges of the V-length segment, cf. exercise 1 in [7, Chapt. 3], exercise 4b in [7, Chapt. 5], and Section B in [7, Chapt. 11]). For example, if the data is bounded by M, $|x(t)| \leqslant M$, and $a(t) \geqslant 0$, then it can be shown that the error in this approximation is worst-case bounded by $r_a(\tau)\, M^2 T/V$. The first and last equalities above are simply applications of the cyclic-periodogram/cyclic-correlogram relation first established in 7 [~~1~~, Chapter 11] together with the convolution theorem (which is used in the last equality).

Interpretation

The left-most member of the above string of equalities (and an approximation) is a biperiodogram of tapered data seen through a sliding window of length T and time-averaged over a window of length V. If this average is discretized, then we are averaging a finite number of biperiodograms of overlapping subsegments over the V-length data record. (It is fairly well known that little is gained—although nothing but computational efficiency is lost—by overlapping segments more than about 50 percent.) The right-most member of the above string is a biperiodogram of untapered data seen through a window of length V and frequency-smoothed along the anti-diagonal $g = (f_1 + f_2)/2$, using a smoothing window $(1/T)\ |A(g)|^2$, for each fixed diagonal $\alpha = f_1 - f_2$. Therefore, given a V-length segment of data, one obtains approximately the same result, whether one averages biperiodograms on subsegments (TAM) or frequency smoothes one biperiodogram on the undivided segment (FSM). Given V, the choice of T determines both the width of the frequency smoothing windows in FSM and the length of the subsegments in TAM. Given V and choosing $T << V$, one can choose either of these two methods and obtain approximately the same result (barring outliers within T of the edges of the data segment of length V). By choosing $f_1 = f_2$ (i.e., $\alpha = 0$), we see the biperiodograms reduce to the more common periodograms, and the equivalence then applies to methods of estimation of power spectral densities, rather than bispectra. Bispectra are also called cyclic spectral densities and spectral correlation functions [7]. As first proved in [7], the FSM and TAM spectral correlation measurements converge to exactly the same quantity, namely, the limit spectral correlation function (when it exists), in the limit as $V \to \infty$ and $T \to \infty$, in this order. Further, this limit spectral correlation function, also called the limit cyclic spectral density, is equal to the Fourier transform of the limit cyclic autocorrelation, as first proved in [7], where this relation is called the *cyclic Wiener relation* because it generalizes the Wiener relation between the PSD and autocorrelation from $\alpha = 0$ to $\alpha \neq 0$:

$$S_x^\alpha(f) = \int_{-\infty}^{\infty} R_x^\alpha(\tau) e^{-i2\pi f\tau} d\tau$$

where

$$R_x^\alpha(\tau) \stackrel{\Delta}{=} \lim_{T\to\infty} R_a^\alpha(t,\tau)$$

$$S_x^\alpha(f) \stackrel{\Delta}{=} \lim_{T\to\infty} \lim_{V\to\infty} \frac{1}{V} \int_{-V/2}^{V/2} S_a(t-u; f_1, f_2) du$$

with $\alpha = f_1 - f_2$.

In the special circumstance where $T << V$ cannot be satisfied because of the degree of spectral resolution (smallness of $1/T$) that is required, there is no known general and provable argument that either method is superior to the other. It has been argued that, since the TAM involves time averaging, it is less appropriate than the FSM for nonstationary data. The results presented here, however, show that, for $T << V$, neither the TAM nor the FSM is more appropriate than the other for nonstationary data. And, when $T << V$ is not satisfied, there is no known evidence that favors either method for nonstationary data.

The derivation of the approximation between the FSM and TAM presented here uses a continuous-time model. However, a completely analogous derivation of an approximation between the discrete-time FSM and TAM is easily constructed. When the spectral correlation function is being measured for many values of the frequency-separation parameter, α, the TAM, modified to what is called the FFT accumulation method (FAM), is much more computationally efficient than the FSM implemented with an FFT [12].

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