IV. BASIC CONCEPTS AND DEFINITIONS (AND SOME PHILOSOPHY)

We need to understand the similarities and differences between the stochastic-process approach and the non-stochastic time-series approach to conceptualizing and defining/modeling

stationary (S),

cyclostationary (CS), and

polycyclostationary (PCS)

signals.

CS $\Longrightarrow$ one period

PCS $\Longrightarrow$ multiple periods

AESTHETICS AND UTILITY

The nonstochastic time-series approach to this subject

1) has been criticized by some mathematicians (e.g., information theorists and probabilists) for its lack of mathematical rigor*,

2) has been both criticized and praised by engineers and scientists for its disregard for the orthodox, or conventional, or just popular.

The aim in developing this approach has been to bring the aesthetics of mathematics and the utility of engineering together to produce "elegant problem solving".

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* I do not believe this is an inherent weakness in the approach, but rather a result of insufficient mathematical effort.

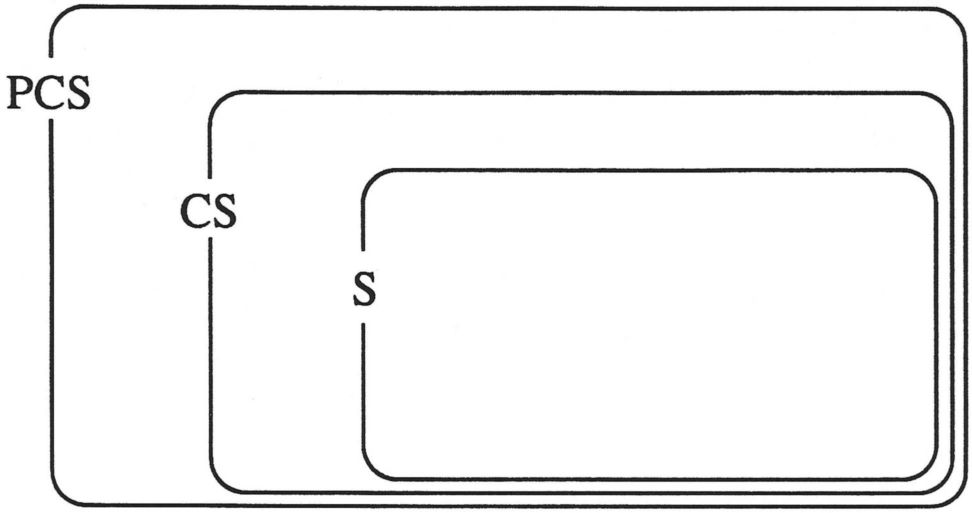
This lecture today aims at the same target: the focusing of attention on important concepts for:

1) mathematicians who care about the applicability of the mathematics of (poly) cyclostationary processes, and

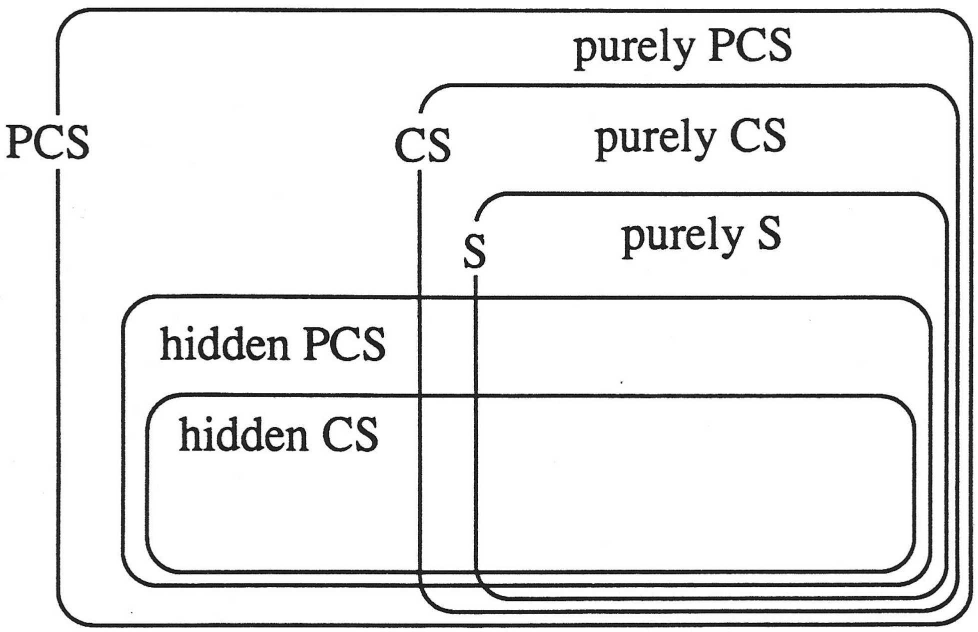
2) engineers who seek more than a superficial understanding of not only the "how" but also the "why" of (poly) cyclostationary signal processing.

SUBCLASSES

Time-Series



Stochastic Processes



STOCHASTIC PROCESSES

Let X(t) be a "well-behaved" stochastic process (with measure μ on the probability space Ω). Consider the event indicator

$$I[X(t) - x] \triangleq \begin{cases} 1, & X(t) < x \\ 0, & X(t) \geq x. \end{cases}$$

The expected value of this event indicator is the probability distribution (PD) function

$$F_{X(t)}(x) \triangleq \text{Prob}\,\{X(t) < x\} = E\{I[X(t) - x]\}$$

where

$$E\{h\} \triangleq \int_{\Omega} h(\omega)\, d\mu(\omega).$$

Therefore, the joint PD function for the set

$$\underline{X}(t) \triangleq \{X(t + t_1), \ldots, X(t + t_n)\}$$

is given by the expectation

$$F_{\underline{X}(t)}(\underline{x}) = E\{\prod_{j=1}^{n} I[X(t + t_j) < x_j]\}$$

and the joint probability density (Pd) function is

$$f_{\underline{X}(t)}(\underline{x}) = \frac{\partial^n}{\partial x_1 \ldots \partial x_n} F_{\underline{X}(t)}(\underline{x})$$

FUNDAMENTAL THEOREM OF EXPECTATION

For any g(·) for which $E\{g[\underline{X}(t)]\}$ exists, we have

$$E\{g[\underline{X}(t)]\} \triangleq \int y \, f_{g[\underline{X}(t)]}(y) \, dy$$

$$= \int g(\underline{x}) \, f_{\underline{X}(t)}(\underline{x}) \, d\underline{x}$$

TIME-SERIES

Let x(t) be a "well-defined" time-series (on the real line $-\infty < t < \infty$). Consider the event indicator

$$I[x(t) - x] \triangleq \begin{cases} 1, & x(t) < x \\ 0, & x(t) \geq x. \end{cases}$$

The time-average of this event indicator is the ***fraction-of-time*** (FOT) PD

$$\hat{F}^0_{x(t)}(x) \triangleq \widehat{\text{Prob}}\{x(t) < x\} = \hat{E}^0\{I[x(t) - x]\}$$

where

$$\hat{E}^0\{h(t)\} \triangleq \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} h(t + t')\, dt' .$$

Therefore the joint FOT PD is given by

$$\hat{F}^0_{\underline{x}(t)}(\underline{x}) = \hat{E}^0\{\prod_{j=1}^{n} I[x(t + t_j) < x_j]\}$$

and the joint FOT Pd is

$$\hat{f}^0_{\underline{x}(t)}(\underline{x}) = \frac{\partial^n}{\partial x_1 \ldots \partial x_n} \hat{F}^0_{\underline{x}(t)}(\underline{x})$$

FUNDAMENTAL THEOREM OF TIME-AVERAGING

For every $g(\cdot)$ for which $\hat{E}^0\{g[\underline{x}(t)]\}$ exists, we have

$$\hat{E}^0\{g[\underline{x}(t)]\} \triangleq \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} g[x(t+t')]\, dt'$$

$$= \int g(\underline{x})\, \hat{f}^{\,0}_{\underline{x}(t)}(\underline{x})\, d\underline{x}$$

$\hat{E}^0\{\cdot\}$ = ***constant component extractor***

= ***temporal expectation***

DUALITY

We can see that there is a ***duality*** between the ***probability-space*** theory of stochastic processes and the ***time-space*** theory of time-series.

Wold [1948] tried to formalize this in an isomorphism based on the mapping

$$x(t + \sigma) \rightarrow X(t, \omega(\sigma))$$

where $X(t, \omega)$ is a sample path of the stochastic process. That is, the ensemble members of $X(t)$ correspond to translates of $x(t)$ in this isomorphism.

While this isomorphism is conceptually useful, a mathematically rigorous study of it has not (to my knowledge) been performed (existence of stochastic process?).

VIABILITY OF TIME-SPACE THEORY

1) Do "well-behaved" time-series models exist?

   Yes; consider any typical sample path of any ergodic stochastic process.

2) Can we construct useful time-series models?

   Yes, in the same way we do for stochastic-process models, except we specify $\hat{F}^{0}_{\underline{x}(t)}(\underline{x})$ instead of $F_{\underline{X}(t)}(\underline{x})$.

3) Does this reliance, of existence, on stochastic processes detract from the conceptual simplicity of working with time-series rather than stochastic processes?

   In my opinion, no.

Let us trace the paths for both stochastic processes and time-series so that we can see specifically where they are parallel and where they diverge.

STOCHASTIC PROCESSES

Probability-Space Definitions (for order n)

Stationary (S) process:

$F_{\underline{X}(t)}(\underline{x})$ is independent of t

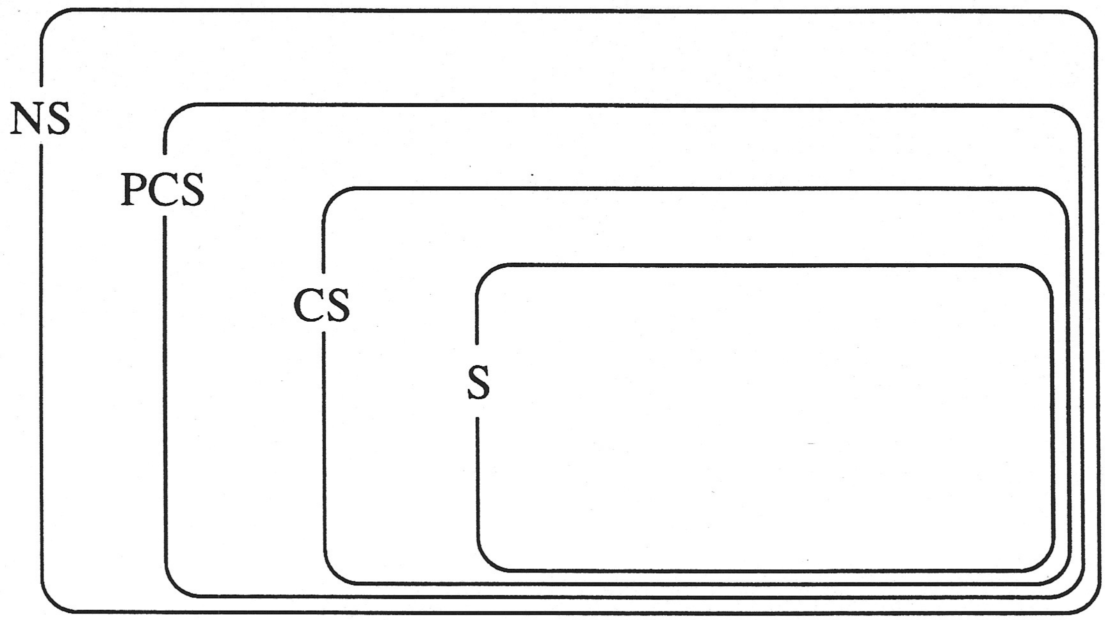
Cyclostationary (CS) process with period T:

$F_{\underline{X}(t)}(\underline{x})$ is periodic in t with period T

Polycyclostationary (PCS) process with periods $\{T\} = T_1, T_2, T_3, \ldots$:

$F_{\underline{X}(t)}(\underline{x})$ is polyperiodic in t with periods $\{T\}$ (which is a sum of periodic functions with single periods $T_1, T_2, T_3, \ldots$)

STOCHASTIC PROCESSES



STOCHASTIC PROCESSES

Polyperiodic PD

$$F_{\underline{X}(t)}(\underline{x}) = \sum_{\alpha} F^{\alpha}_{\underline{X}(0)}(\underline{x})e^{i2\pi\alpha t} = \sum_{\alpha} F^{\alpha}_{\underline{X}(t)}(\underline{x})$$

where

$$F^{\alpha}_{\underline{X}(t)}(\underline{x}) \triangleq \lim_{Z\to\infty} \frac{1}{Z}\int_{-Z/2}^{Z/2} F_{\underline{X}(t+t')}(\underline{x})e^{-i2\pi\alpha t'}\,dt'$$

$$= \hat{E}^{0}\left\{F_{\underline{X}(t)}(\underline{x})e^{-i2\pi\alpha t}\right\}$$

$$\triangleq \hat{E}^{\alpha}\left\{F_{\underline{X}(t)}(\underline{x})\right\}$$

$$\hat{E}^{\alpha}\{\cdot\} \triangleq \hat{E}^{0}\{(\cdot)e^{-i2\pi\alpha t}\}$$

= ***sine-wave component extractor***

TIME-SERIES

Before we can give the dual time-space definitions of S, CS, and PCS, we need to generalize the temporal expectation $\hat{E}^{0}\{\cdot\}$.

$$\hat{E}^{\{\alpha\}}\{\cdot\} \triangleq \sum_{\alpha \in \{\alpha\}} \hat{E}^{\alpha}\{\cdot\}$$

= ***multiple sine-wave component extractor***

= ***polyperiodic component extractor***

$$\alpha\text{'s} = \text{harmonics of } \frac{1}{T_1}, \frac{1}{T_2}, \frac{1}{T_3}, \cdots$$

TIME-SERIES

Polyperiodic FOT PD:

$$\hat{F}^{\{\alpha\}}_{\underline{x}(t)}(\underline{x}) = \hat{E}^{\{\alpha\}}\{\prod_{j=1}^{n} I[x(t+t_j) - x_j]\}$$

Polyperiodic FOT Pd:

$$\hat{f}^{\{\alpha\}}_{\underline{x}(t)}(\underline{x}) = \frac{\partial^n}{\partial x_1 \ldots \partial x_n} \hat{F}^{\{\alpha\}}_{\underline{x}(t)}(\underline{x})$$

FUNDAMENTAL THEOREM OF POLYPERIODIC-COMPONENT EXTRACTION

For every $g(\cdot)$ for which $\hat{E}^{\{\alpha\}}\{g[\underline{x}(t)]\}$ exists, we have

$$\hat{E}^{\{\alpha\}}\{g[\underline{x}(t)]\} \triangleq \sum_{\alpha} \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} g[(\underline{x}(t+t')]e^{-i2\pi\alpha t'}\, dt'$$

$$= \int g(\underline{x})\, \hat{f}^{\{\alpha\}}_{\underline{x}(t)}(\underline{x})\, d\underline{x}$$

TIME-SERIES

Time-Space Definitions (for order n)

Stationary (S) time-series:

$\hat{F}^{\{\alpha\}}_{\underline{x}(t)}(\underline{x})$ exists and $\not\equiv 0$ and is independent of t ($\{\alpha\} = \{0\}$)

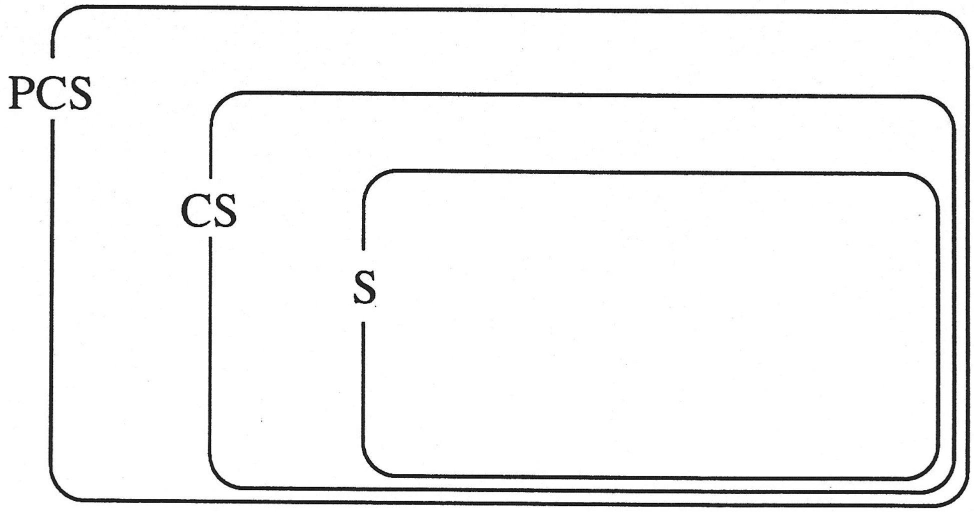
Cyclostationary (CS) time-series with period T:

$\hat{F}^{\{\alpha\}}_{\underline{x}(t)}(\underline{x})$ exists and $\not\equiv 0$ and is periodic in t with period T ($\{\alpha\}$ = harmonics of 1/T)

Polycyclostationary (PCS) time-series with periods $\{T\} = T_1, T_2, T_3, \ldots$:

$\hat{F}^{\{\alpha\}}_{\underline{x}(t)}(\underline{x})$ exists and $\not\equiv 0$ and is polyperiodic in t with periods $\{T\}$)

TIME-SERIES



TIME-SERIES

This represents a modification of previous terminology

Previous:

stationary (Wold, 1940s)—

$\hat{F}^{0}_{\underline{x}(t)}(\underline{x})$ exists and $\not\equiv 0$

purely stationary (Gardner, 1980s)—

stationary and $\hat{F}^{\alpha}_{\underline{x}(t)}(\underline{x}) \equiv 0$ for $\alpha \neq 0$

cyclostationary (T) (Gardner)—

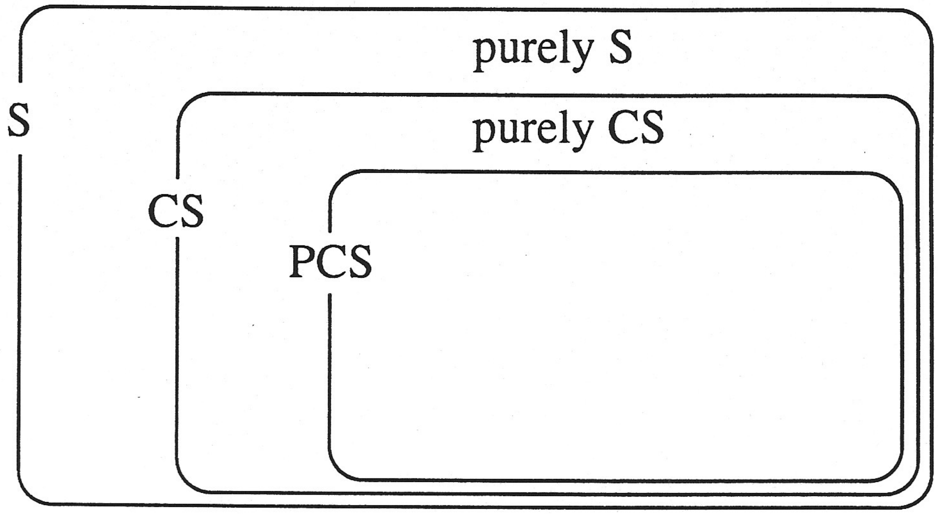
$\hat{F}^{\{\alpha\}}_{\underline{x}(t)}(\underline{x})$ exists and $\not\equiv 0$ for some $\{\alpha\}$ = {harmonics of 1/T}

purely cyclostationary (T) (Gardner)—

cyclostationary and $\hat{F}^{\{\alpha\}}_{\underline{x}(t)}(\underline{x}) \equiv 0$ for $\alpha \notin$ {harmonics of 1/T}

PREVIOUS SUBCLASSES

Time-Series



Inverted Nesting

purely S = S – CS

purely CS = CS – PCS

STOCHASTIC PROCESSES

Cycloergodicity

X(t) is ***ergodic*** (E) iff: for every g(·) for which E{g[$\underline{X}$(t)]} exists, we have

$$\hat{E}^0\{E\{g[\underline{X}(t)]\}\} = \hat{E}^0\{g[\underline{X}(t)]\} \quad \text{w.p. } 1.$$

For a stationary process, $\hat{E}^0\{E\{\cdot\}\} = E\{\cdot\}$.

X(t) is ***cycloergodic*** (CE) with period T iff: for every g(·) for which E{g[$\underline{X}$(t)]} exists, we have (with {α} = {harmonics of 1/T}

$$\hat{E}^{\{\alpha\}}\{E\{g[\underline{X}(t)]\}\} = \hat{E}^{\{\alpha\}}\{g[\underline{X}(t)]\} \quad \text{w.p. } 1.$$

For a CS process, $\hat{E}^{\{\alpha\}}\{E\{\cdot\}\} = E\{\cdot\}$.

X(t) is ***polycycloergodic*** (PCE) with periods {T} iff it is cycloergodic with period T_k for k = 1, 2, 3, … .

STOCHASTIC PROCESSES

Hidden cyclostationarity in stochastic-process models

If X(t) is S and PCE with all periods, then its sample paths are stationary time-series (w.p. 1).

If X(t) is S (and possibly E) but not CE, its sample paths can be CS (w.p. 1).

If X(t) is CS and PCE with all periods, then its sample paths are CS time-series (w.p. 1).

If X(t) is CS (and possibly CE), but not PCE, its sample paths can be PCS (w.p. 1).

Such non-CE and non-PCE models typically result from the (explicit or implicit) inclusion of random-phase variables in the stochastic process model.

STOCHASTIC PROCESSES

Refined probability-space definitions

If X(t) is S and PCE with all periods, then it is defined to be ***purely stationary***, and its sample paths are stationary time-series (w.p. 1): there is no hidden CS.

If X(t) is CS and PCE with all periods, then it is defined to be ***purely CS***, and its sample paths are purely CS time-series (w.p. 1): there is no hidden PCS.

EXAMPLE

Let

$$X(t) = A(t) + B(t)\cos(\omega_1 t + \theta_1) + C(t)\cos(\omega_2 t + \theta_2),$$

where A(t), B(t), and C(t) are purely stationary ergodic processes.

STOCHASTIC PROCESSES

If θ_1 and θ_2 are non-random, then X(t) is PCS and PCE.

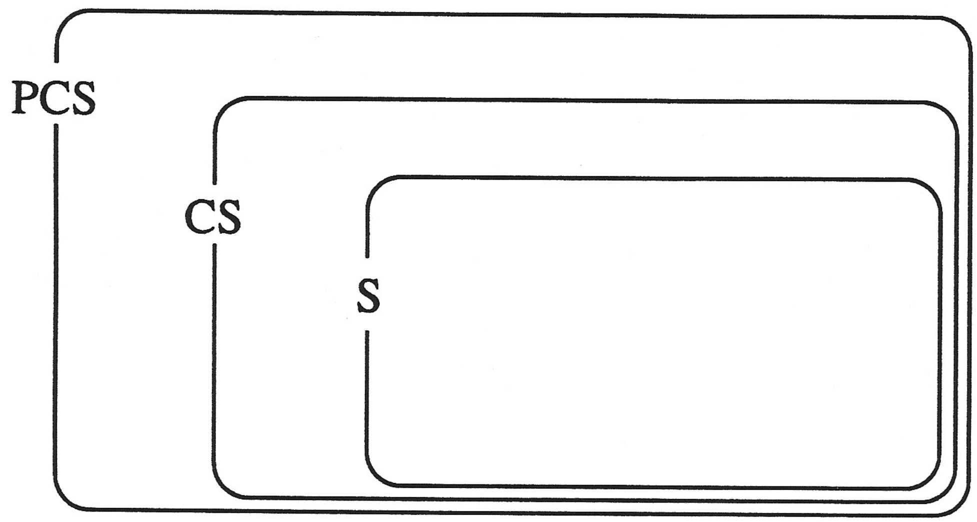
If θ_1 and/or θ_2 is random, then (depending on their PDs) X(t) can be PCS (with periods T_1 and T_2), or it can be CS (with period T_1 or T_2), or it can be S, and X(t) is not PCE.

TIME-SERIES

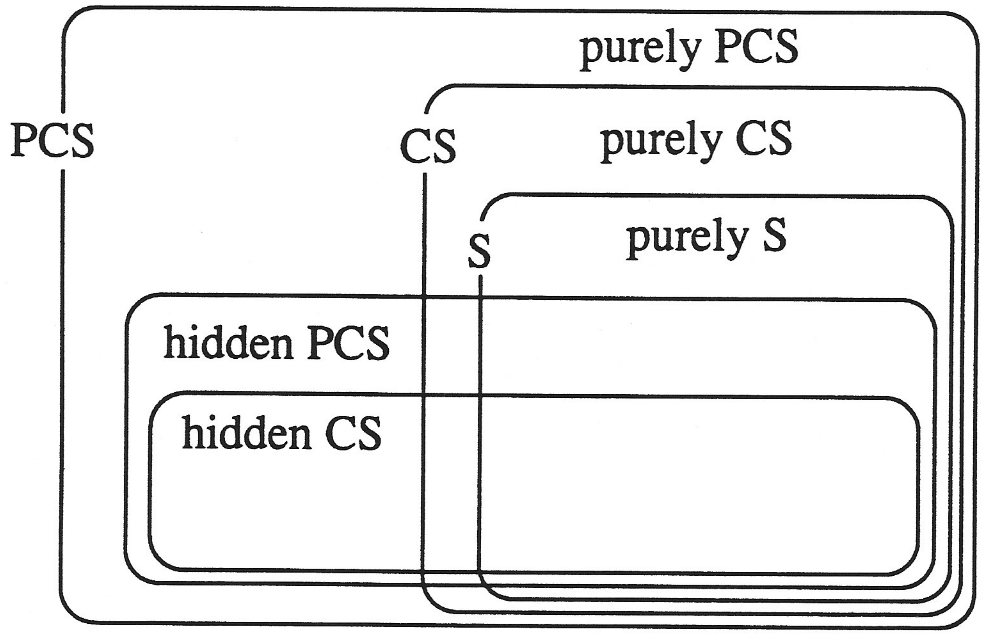
With probability one, the sample paths of X(t) are PCS time-series.

SUMMARY OF SUBCLASSES

Time-Series



Stochastic Processes



PHASE RANDOMIZATION

Even if θ_1 and θ_2 are both nonrandom, we can introduce θ into $X(t)$ to obtain

$$Y(t) = X(t + \theta)$$

which can be changed from PCS to CS or to S by choice of the distribution for θ.

NONUNIQUENESS OF MODELS

Stochastic Processes:

We can change the stochastic process from

$$PCS \rightarrow CS \rightarrow S$$

by phase-randomization with a single phase variable:

$$X(t) \rightarrow X(t + \theta)$$

Stochastic Processes (and Time-Series):

We can change the PD function from

polyperiodic $\rightarrow$ periodic $\rightarrow$ constant

by time averaging; e.g.,

$$\hat{E}^0\{\hat{F}^{\{\alpha\}}_{\underline{x}(t)}\} = \hat{F}^0_{\underline{x}(t)}$$

Both time-averaging and phase randomization result in hidden cyclostationarity.

PITFALLS OF NONUNIQUE MODELS

Hidden Statistical Dependence

Let SI = Statistical Independence (e.g., of two variables)

1) SI in CS model $\not\Rightarrow$ SI in S model
2) SI in S model $\not\Rightarrow$ SI in CS model

Proof of 1)

$$f_{X_1(t),X_2(t)} = f_{X_1(t)} f_{X_2(t)}$$

$$\not\Rightarrow \hat{E}^0\{f_{X_1(t),X_2(t)}\} = \hat{E}^0\{f_{X_1(t)}\}\, \hat{E}^0\{f_{X_2(t)}\}$$

Proof of 2) — by example

$$X_1(t) = Z(t) = \text{i.i.d.} \pm 1$$

$$X_2(t) = Z(t)\cos(t)$$

$$\hat{E}^0 E\left\{X_1^n(t) X_2^m(t)\right\} = \hat{E}^0 E\left\{X_1^n(t)\right\} \hat{E}^0 E\left\{X_2^m(t)\right\}$$

for all n, m, but

$$E\{X_1^n(t) X_2^m(t)\} \neq E\{X_1^n(t)\}\, E\{X_2^m(t)\}$$

for n and m odd.

TAKING STOCK

One Conclusion:

When a process is not PCE, the hidden CS or hidden PCS can result in single-sample-path behavior (w.p. 1) that cannot be predicted from probabilistic analysis (unless the hidden CS can be revealed by conditioning on certain random phase variables).

An Important Fact:

The theory of PCE is mostly nonexistent and appears to require nontrivial extensions/ generalizations of the theory of E and the incomplete theory of CE.

Another Important Fact:

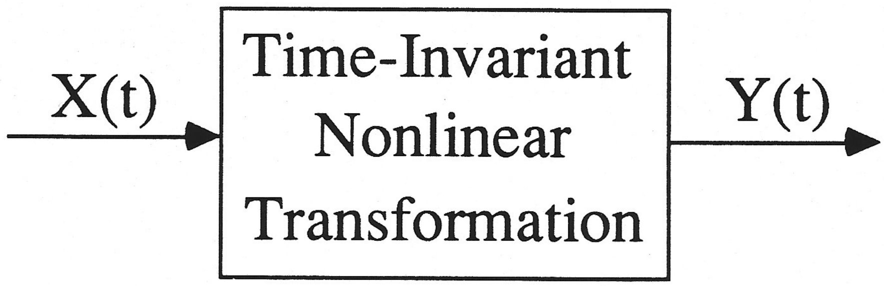
The commonplace approach, to deriving ad hoc signal-processing algorithms, of replacing expectation operations $E\{\cdot\}$ in analytical expressions with time-average operations $\hat{E}^0\{\cdot\}$, or (when both operations are present in the analytical expression) of deleting the expectation operation, cannot be justified (and will often fail to produce the desired results) when the stochastic-process model used is not PCE.

A Related Important Fact:

The "optimum" solutions to inference and decision problems (e.g., for signal estimation and detection) that are based on S and E, but not CE, (or based on CS and CE, but not PCE), process models can be highly inferior to inference and decision rules that exploit the hidden CS (or hidden PCS).

EXAMPLES

1)



X(t) is S and E, but not CE, with no spectral lines.

Y(t) is S and E but, because of the hidden CS in X(t), contains spectral lines. The presence of these spectral lines cannot be explained except by virtue of the hidden CS in X(t).

2) X(t) is S and E, but not CE, and admits an exact AR model (with white residuals).

But the sample paths of the residuals are still partially predictable (w.p. 1) using periodic predictors derived from the sample-path statistics.

Furthermore, the sample paths of X(t) can also admit periodic AR (PAR) models with asymptotically white residuals (w.p. 1) that are unpredictable (w.p. 1) and with periodic order that at times within the period can be less than that of the AR model.

3) X(t) and Y(t) are jointly S and E, but not CE, and, according to the usual definition of ***causality***, there is no causal relationship of X(t) to Y(t). That is, no linear or nonlinear ***time-invariant*** operation on X(t) and its past has any prediction capability for Y(t) and its future.

Yet, each sample path of Y(t) can possibly be perfectly ***cyclically caused*** by the corresponding sample path of X(t). That is, a periodic operation on X(t) can possibly perfectly predict Y(t).

EXAMPLE

$X(t) = Z(t) = \pm 1$

$Y(t) = Z(t - \tau)\cos(t + \theta)$; θ uniform on $[0, 2\pi]$

$E\{X^n(t-\mu)Y^m(t)\} = E\{X^n(t-\mu)\}E\{Y^m(t)\}$

for all μ.

But

$Y(t) = X(t - \tau)\cos(t + \theta)$

4) $Z(t) = X(t) + Y(t)$, where $X(t)$ and $Y(t)$ are statistically independent, S, and E, but not CE, processes that have identical spectral densities. The Wiener filter for extracting $X(t)$ from $Z(t)$ (separating $X(t)$ and $Y(t)$) is essentially useless.

Yet the sample paths of $X(t)$ and $Y(t)$ can possibly be perfectly separated with a periodic filter.

Examples include communication signals such as

digital QAM
AM
PSK
ASK
PAM

SPECIFIC EXAMPLE:

N spectrally coincident digital QAM signals with excess bandwidth $\geq (N-1)100\%$ can be perfectly separated

5) $\underline{X}(t) = \{X_1(t), X_2(t)\}$ is purely S with a probabilistic model that is very similar to that of $\underline{Y}(t) = \{Y_1(t), Y_2(t)\}$, which is CS (e.g., $\underline{X}(t)$ and $\underline{Y}(t)$ are both Gaussian processes and the PSDs of $\underline{X}(t)$ equal those of the stationarized $\underline{Y}(t)$). The Cramér-Rao bounds of the same parameters in each of $\underline{X}(t)$ and $\underline{Y}(t)$ (e.g., the TDOA) can be drastically different.

This has been demonstrated for TDOA at two reception platforms and for AOA at a sensor array.

Moreover, even the Cramér-Rao bound of the stationarized $\underline{Y}(t)$ can be drastically different from that of the purely S $\underline{X}(t)$.

6) X(t) and Y(t) are independent, S, and E, but not CE, and

under hypothesis 1: Z(t) = X(t) + Y(t)

under hypothesis 2: Z(t) = Y(t)

The "optimum" (e.g., max posterior prob.) detector for the presence of X(t) in Z(t) can be greatly outperformed by detectors that exploit the hidden CS in X(t) and/or Y(t) (e.g., joint max posterior prob. detector and phase estimator).

ONE APPROACH TO THIS SITUATION

(For Mathematicians)

Take **PATH 1**: Develop the needed theory of PCE.

Current status:

Substantial progress has been made for:

1) CE w.p. 1 for discrete-time CS and Gaussian continuous-time CS;
2) PCE in m.s. for finite-order moments of discrete- and continuous-time PCS.

Little or no progress has been made for:

1) PCE w.p. 1 for discrete-time PCS;
2) CE w.p. 1 for non-Gaussian continuous-time CS and PCS, and
3) PCE w.p. 1 for continuous-time CS and PCS.

A CHALLENGE TO MATHEMATICIANS

The only paper to address PCE w.p. 1 [Boyles and Gardner, 1983] suggests that a substantial breakthrough will be required (even for the much less technical case of discrete time): conventional approaches and ideas apparently lead to dead ends.

I see a challenge not unlike that Birkhoff faced around 1930 when he formulated and proved the fundamental ***ergodic theorem*** to replace the very unsatisfying "ergodic hypothesis".

We need a fundamental ***polycycloergodic theorem*** that elegantly formalizes our informal notion of a PCE process in terms of a necessary and sufficient condition on the associated probability measure.

A PROPOSITION

The most useful concept we have for applications is the following ***unproved*** proposition:

> *A PCS process constructed from stable (decaying-memory) nonrandom polyperiodic transformations of purely stationary ergodic processes are PCE.*

Stochastic-process models for many, if not most, communication signals can be constructed in this way.

ADVANTAGES OF THE STOCHASTIC-PROCESS APPROACH

1) It is the orthodox approach to modeling and studying evolutionary random phenomena and it is, therefore, attractive to those already familiar with it.

2) Mathematicians do know how, in principle, to construct stochastic process models from elementary mathematical constructs (Borel fields, sigma algebras, probability measures, etc.)

Therefore, there is a greater likelihood of success in constructing a ***mathematical*** theory of PCS and PCE processes from a few basic axioms.

3) It *is* possible, in principle, to exploit the hidden CS in a non-CE process within the conventional framework of stochastic processes.

But, this requires that one have a model of the hidden CS that is ***explicitly dependent*** on a random phase variable θ that is responsible for the lack of CE so that one can calculate probability densities and expectations ***conditioned on θ***.

4) Development of the theory of PCE will pave the way for making the time-space theory mathematically rigorous.

AN ALTERNATIVE APPROACH

(For Pragmatic Engineers)

Perspective:

The probability-space approach based on expectation introduces abstractions that, in many applications (e.g., many problems for which single-sample-path signal-processing is of interest), have no redeeming ***practical*** value.

Some of these abstractions can be properly dealt with only with a theory of PCE that is presently nonexistent. Regressing back to pre-1930 and adopting a "PCE hypothesis" is very unappealing (because the hypothesis can be false).

Response: Take **PATH 2—**

Adopt the time-space approach whose theory is in many ways dual to that of the probability-space approach, but without the practical drawbacks associated with cycloergodicity and the distracting abstraction associated with expectation over ensembles.

THE ESSENCE OF CYCLOSTATIONARITY

The essence of cyclostationarity is the fact that sinewaves can be generated from random data by applying certain nonlinear transformations.

The time-space theory of cyclostationarity arises ***naturally*** out of the fundamental theorem of sine-wave component extraction using $\hat{E}^{\{\alpha\}}$.

The expectation E that gives rise to the probability-space theory has little to do with the essence of cyclostationarity.

BUT, CAN WE CONSTRUCT TIME-SERIES MODELS?

Yes. Time-series models for many, if not most, communication signals can be constructed by subjecting one or more elementary time-series (e.g., purely stationary "white") to elementary transformations such as filters, periodic modulators, multiplexors, etc.

Defining Properties of a Discrete-Time Purely Stationary White Time-Series

1) Whiteness: $\hat{f}^{0}_{\underline{x}(t)}(\underline{x})$

$$= \hat{f}^{0}_{x(t+t_1)}(x_1)\ \hat{f}^{0}_{x(t+t_2)}(x_2)\ \ldots\ \hat{f}^{0}_{x(t+t_n)}(x_n)$$

for unequal $t_1, t_2, \ldots, t_n$.

2) Pure stationarity: $\hat{f}^{\{\alpha\}}_{x(t)}(x) \equiv \hat{f}^{0}_{x(t)}(x)$ for all $\{\alpha\}$.

3) Existence: sample path of i.i.d. stochastic process.

CAN WE DO PROBABILISTIC ANALYSIS USING TIME-SPACE THEORY?

Yes. Performance measures such as bias, variance, Cramér-Rao bounds, confidence intervals, probabilities of decision-errors, etc., can be calculated using time-space theory just as well as they can using probability-space theory.

A CURRENT ASSESSMENT

1) Considerable progress in the development and application of the time-space (or ***temporal-probability***, or ***fraction-of-time probability***) theory of CS and PCS time-series has been made since its adoption by the UCD group in 1985. This includes:

   a) Temporal and spectral second-order-moment theory (cyclic autocorrelation and cyclic spectra, or spectral correlation functions)

   b) Temporal and spectral higher-order-moment and cumulant theory (cyclic cumulants and cyclic polyspectra, or spectral cumulants)

   c) The rudiments of fraction-of-time probability distribution theory

   d) A wide variety of applications of the theory to signal processing and communications problems

FURTHER SUPPORT FOR PATH 2

1) The temporal-probability approach, which is centered on the concrete sine-wave extraction operation, has led naturally to a ***derivation*** of the cumulant as the solution to a fundamental problem in characterizing higher-order CS and PCS.

   It is doubtful that this derivation would have have been discovered within the stochastic-process framework, which is centered on the abstract expectation operation.

   This derivation will be discussed by Chad Spooner tomorrow afternoon.

2) The conceptual gap between the existing time-space theory and its application is perceived by its current users (at UCD) to be much narrower than it is for the dual probability-space theory.

CONJECTURE

For every theorem that can be proved for a PCE PCS process, a dual theorem can be proved for a PCS time-series—and vice-versa. (Generalization of Wold's isomorphism from S to PCS)

EXISTENCE

There does remain a fundamental question that is not yet always answerable:

Given $\hat{F}^{\{\alpha\}}_{\underline{x}(t)}$ for all n does there exist a corresponding x(t)?

We have sufficient conditions on $\hat{F}^{\{\alpha\}}_{\underline{x}(t)}$ that guarantee existence of x(t): they are identical to the conditions that guarantee that $F_{\underline{X}(t)}$ is PCE (mixing conditions).

But we do not yet have a ***necessary*** and sufficient condition. (Another challenge for mathematicians)

CONCLUSIONS

1) The more abstract theory of PCS stochastic processes will undoubtedly be found to be of considerable value as it is developed, and those who are so inclined are encouraged to pursue this approach.

2) The less abstract theory of cyclostationary time-series is more accessible to the engineer interested in theory as a ***conceptual aid for solving practical problems***. It should be the preferred approach (not only at UCD).

   The practical value of this approach is amply demonstrated for parametric and nonparametric spectral analysis of S as well as CS and PCS time-series in my 1987 book.

3) Both theories present important challenges to mathematicians.