What is Mathematics?

The Most Misunderstood Subject

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For more than two thousand years, mathematics has been a part of the human search for understanding. Mathematical discoveries have come both from the attempt to describe the natural world and from the desire to arrive at a form of inescapable truth from careful reasoning. These remain fruitful and important motivations for mathematical thinking, but in the last century mathematics has been successfully applied to many other aspects of the human world: voting trends in politics, the dating of ancient artifacts, the analysis of automobile traffic patterns, and long-term strategies for the sustainable harvest of deciduous forests, to mention a few. Today, mathematics as a mode of thought and expression is more valuable than ever before. Learning to think in mathematical terms is an essential part of becoming a liberally educated person.
-- Kenyon College Math Department Web Page

"An essential part of becoming a liberally educated person?" Sadly, many people in America, indeed, I would have to say very many people in America, would find that a difficult and puzzling concept. The majority of educated Americans do not think of Mathematics when they think of a liberal education. Mathematics as essential for science, yes, for business and accounting, sure, but for a liberal education?

Why do so many people have such misconceptions about mathematics?

The great misconception about mathematics -- and it stifles and thwarts more students than any other single thing -- is the notion that mathematics is about formulas and cranking out computations. It is the unconsciously held delusion that mathematics is a set of rules and formulas that have been worked out by God knows who for God knows why, and the student's duty is to memorize all this stuff. Such students seem to feel that sometime in the future their boss will walk into the office and demand "Quick, what's the quadratic formula?" Or, "Hurry, I need to know the derivative of 3x^2 - 6x + 7." There are no such employers.

What is mathematics really like?

Mathematics is not about answers, it's about processes. Let me give a series of parables to try to get to the root of the misconceptions and to try to illuminate what mathematics IS all about. None of these analogies is perfect, but all provide insight.

Scaffolding

When a new building is made, a skeleton of steel struts called the scaffolding is put up first. The workers walk on the scaffolding and use it to hold equipment as they begin the real task of constructing the building. The scaffolding has no use by itself. It would be absurd to just build the scaffolding and then walk away, thinking that something of value has been accomplished.

Yet this is what seems to occur in all too many mathematics classes in high schools. Students learn formulas and how to plug into them. They learn mechanical techniques for solving certain equations or taking derivatives. But all of these things are just the scaffolding. They are necessary and useful, sure, but by themselves they are useless. Doing only the superficial and then thinking that something important has happened is like building only the scaffolding.

The real "building" in the mathematics sense is the true mathematical understanding, the true ability to think, perceive, and analyze mathematically.

Ready for the Big Play

Professional athletes spend hours in gyms working out on equipment of all sorts. Special trainers are hired to advise them on workout schedules. They spend hours running on treadmills. Why do they do that? Are they learning skills necessary for playing their sport, say basketball?

Imagine there are three seconds left in the seventh game of the NBA championship. The score is tied. Time out. The pressure is intense. The coach is huddling with his star players. He says to one, "OK Michael, this is it. You know what to do." And Michael says, "Right coach. Bring in my treadmill!"

Duh! Of course not! But then what was all that treadmill time for? If the treadmill is not seen during the actual game, was it just a waste to use it? Were all those trainers wasting their time? Of course not. It produced (if it was done right!) something of value, namely stamina and aerobic capacity. Those capacities are of enormous value even if they cannot be seen in any immediate sense. So too does mathematics education produce something of value, true mental capacity and the ability to think.

The hostile party goer.

When I was in first grade, we read a series of books about Dick and Jane. There were a lot of sentences like "see Dick run" and so forth. Dick and Jane also had a dog called Spot.

What does that have to do with mathematics education? Well, when I occasionally meet people at parties who learn that I am a mathematician and professor, they sometimes show a bit of repressed hostility. One man once said something to me like, "You know, I had to memorize the quadratic formula in school and I've never once done anything with it. I've since forgotten it. What a waste. Have YOU ever had to use it aside from teaching it?"

I was tempted to say, "No, of course not. So what?" Actually though, as a mathematician and computer programmer I do use it, but rarely. Nonetheless the best answer is indeed, "No, of course not. So what?" and that is not a cynical answer.

After all, if I had been the man's first grade teacher, would he have said, "You know, I can't remember anymore what the name of Dick and Jane's dog was. I've never used the fact that their names were Dick and Jane. Therefore, you wasted my time when I was six years old."

How absurd! Of course, people would never say that. Why? Because they understand intuitively that the details of the story were not the point. The point was to learn to read! Learning to read opens vast new vistas of understanding and leads to all sorts of other competencies. The same thing is true of mathematics. Had the man's mathematics education been a good one he would have seen intuitively what the real point of it all was.

[Added January 4, 2011] Sadly, the memorizing-vs-understanding problem comes up all the time in mathematics classes. Many times, in teaching, say, the Chain Rule in Calculus, I see students who just want me to tell them what to memorize. They don't see why I insist that they understand it. That's like refusing to learn to read in first grade, and just wanting to memorize the name of the dog.

The considerate piano teacher.

Imagine a piano teacher who gets the bright idea that she will make learning the piano "simpler" by plugging up the student's ears with cotton. The student can hear nothing. No distractions that way! The poor student sits down in front of the piano and is told to press certain keys in a certain order. There is endless memorizing of "notes" A, B, C, etc. The student has to memorize strange symbols on paper and rules of writing them. And all the while the student hears nothing! No music! The teacher thinks she is doing the student a favor by eliminating the unnecessary distraction of the sound!

Of course, the above scenario is preposterous. Such "instruction" would be torture. No teacher would ever dream of such a thing, of removing the heart and soul of the whole experience, of removing the music. And yet that is exactly what has happened in most high school mathematics classes over the last 25 years. For whatever misguided reason, mathematics students have been deprived of the heart and soul of the course and been left with a torturous outer shell. The prime example is the gutting of geometry courses, where proofs have been removed or de-emphasized. Apparently, some teachers think that this is "doing the students a favor." Or is it that many teachers do not really understand the mathematics at all?

Step high.

A long time ago when I was in graduate school, the physical fitness craze was starting. A doctor named Cooper wrote a book on Aerobics in which he outlined programs one could follow to build up aerobic capacity, and therefore cardiovascular health. You could do it via running, walking, swimming, stair climbing, or stationary running. In each case, he outlined a week by week schedule. The goal was to work up to what he called 30 "points" per week of exercise during a twelve-week program.

Since it was winter and I lived in a snowy place, I decided to do stationary running. I built a foam padded platform to jog in place. Day after day I would follow the schedule, jogging in place while watching television. I dreamed of the spring when I would joyfully demonstrate my new health by running a mile in 8 minutes, which was said to be equivalent to 30-points-per-week cardiovascular health.

The great day came. I started running at what I thought was a moderate pace. But within a minute I was feeling winded! The other people with me started getting far ahead. I tried to keep up, but soon I was panting, gasping for breath. I had to give up after half a mile! I was crushed. What could have gone wrong? I cursed that darn Dr. Cooper and his book.

I eventually figured it out. In the description of stationary running, it said that every part of one's foot must be lifted a certain distance from the floor, maybe it was 10 inches. In all those weeks, I never really paid attention to that. Someone then checked me, and I wasn't even close to 10 inches. No wonder it had failed! I was so discouraged; it was years before I tried exercising again.

What does that have to do with mathematics education? Unfortunately, a great deal. In the absence of a real test (for me, actually running on a track) it is easy to think one is progressing if one follows well intentioned but basically artificial guidelines. It is all too easy to slip in some way (as I did by not stepping high enough) and be lulled into false confidence. Then when the real test finally comes, and the illusion of competence is painfully shattered, it is all too easy to feel betrayed or to "blame the messenger."

The "real test" I am speaking of is not just what happens to so many high school graduates when they meet freshman mathematics courses. It is that we in the U. S. are falling farther and farther behind most other countries in the world, not just the well-known ones like China, India, and Japan. The bar must be raised, yes, but not in artificial ways, in true, authentic ones.

Cargo cult education.

During World War II in the Pacific Ocean American forces hopped from island to island relentlessly pushing westward toward Japan. Many of these islands in the south Pacific were inhabited by people who had never seen Westerners; maybe their ancestors' years before had left legends of large wooden ships. We can only imagine their surprise and shock when large naval vessels arrived and troops set up communication bases and runways. Airplanes and those who flew them seemed like gods. It seemed to the natives that the men in the radio buildings, with their microphones, radios and large antennas, had the power to call in the gods. All of the things brought by the navy, radios, buildings, food, weapons, furniture, etc. were collectively referred to as "cargo".

Then suddenly the war ended and the Westerners left. No more ships. No more airplanes. All that was left were some abandoned buildings and rusting furniture. But a curious thing happened. The natives on some islands figured that they, too, could call in the gods. They would simply do what the Americans had done. They entered the abandoned buildings, erected a large bamboo pole to be the "antenna", found some old boxes to be the "radio", used a coconut shell to be the "microphone." They spoke into the "microphone" and implored the airplanes to land. But of course, nothing came (except, eventually, some anthropologists!) The practice came to be known as a "Cargo Cult."

The story may seem sad, amusing, or pathetic, but what does that have to do with mathematics education? Unfortunately, a great deal. The south Pacific natives were unable to discern between the superficial outer appearance of what was happening and the deeper reality. They had no understanding that there even exists such a thing as electricity, much less radio waves or aerodynamic theory. They imitated what they saw, and they saw only the superficial.

Sadly, the same thing has happened in far too many high schools in the United States in the last twenty-five years or so in mathematics education. Well meaning "educators" who have no conception of the true nature of mathematics see only its outer shell and imitate it. The result is cargo cult mathematics. They call for the gods, but nothing happens. The cure is not louder calling, it is not more bamboo antennas (i.e. glossy ten-pound text books and fancy calculators). The only cure is genuine understanding of authentic mathematics.

Confusion of Education with Training.

Training is what you do when you learn to operate a lathe or fill out a tax form. It means you learn how to use or operate some kind of machine or system that was produced by people in order to accomplish specific tasks. People often go to training institutes to become certified to operate a machine or perform certain skills. Then they can get jobs that directly involve those specific skills.

Education is very different. Education is not about any particular machine, system, skill, or job. Education is both broader and deeper than training. An education is a deep, complex, and organic representation of reality in the student's mind. It is an image of reality made of concepts, not facts. Concepts that relate to each other, reinforce each other, and illuminate each other. Yet the education is more even than that because it is organic: it will live, evolve, and adapt throughout life.

Education is built up with facts, as a house is with stones. But a collection of facts is no more an education than a heap of stones is a house.

An educated guess is an accurate conclusion that educated people can often "jump to" by synthesizing and extrapolating from their knowledge base. People who are good at the game "Jeopardy" do it all the time when they come up with the right question by piecing together little clues in the answer. But there is no such thing as a "trained guess."

No subject is more essential nor can contribute more to becoming a liberally educated person than mathematics. Become a math major and find out!

So What Good Is It?

Some people may understand all that I've said above but still feel a bit uneasy. After all, there are bills to pay. If mathematics is as I've described it, then perhaps it is no more helpful in establishing a career than, say, philosophy.

Here we mathematicians have the best of both worlds, as there are many careers that open up to people who have studied mathematics. Real Mathematics, the kind I discussed above. See the Careers web page for a sampling.

That brings up one more misconception and one more parable, which I call:

Computers, mathematics, and the chagrinned diner.

About twenty years ago when personal computers were becoming more common in small businesses and private homes, I was having lunch with a few people, and it came up that I was a mathematician. One of the other diners got a funny sort of embarrassed look on her face. I steeled myself for that all too common remark, "Oh I was never any good at math." But no, that wasn't it. It turned out that she was thinking that with computers becoming so accurate, fast, and common, there was no longer any need for mathematicians! She was feeling sorry me, as I would soon be unemployed! Apparently, she thought that a mathematician's work was to crank out arithmetic computations.

Nothing could be farther from the truth. Thinking that computers will obviate the need for mathematicians is like thinking 90 years ago when cars replaced horse drawn wagons, there would be no more need for careful drivers. On the contrary, powerful engines made careful drivers more important than ever.

Today, powerful computers and good software make it possible to use and concretely implement abstract mathematical ideas that have existed for many years. For example, the RSA cryptosystem is widely used on secure internet web pages to encode sensitive information, like credit card numbers. It is based on ideas in algebraic number theory, and its invulnerability to hackers is the result of very advanced ideas in that field.

Finally, here are a few quotes from an essay well worth reading by David R. Garcia on a similar topic:

> *Americans like technology but seldom have a grasp of the science behind it. And the mathematics that is behind the science is regarded as even more mysterious, like an inner sanctum into which only initiates may gain entry. They see the rich and nourishing technological fruit on this tree of knowledge, but they see no deeper than the surface branches and twigs on which these fruits grow. To them, the region behind this exterior of the tree, where the trunk and limbs grow, is pointless and purposeless. "What's the use of math?" is the common query. "I'll never use it." When a nation's leaders are composed primarily of lawyers, administrators, military men and stars of the entertainment industry rather than statesmen, philosophers, the spiritual, and the men and women of science, then it should be no surprise that there is so little grasp of the simple reality that one cannot dispense with the trunk and limbs and still continue to enjoy the fruit.*

..... What is it that would cause us to focus only on this external fruit of material development and play down the antecedent realms of abstraction that lie deeper? It would be good to find a word less condemning than "superficiality", but how else can this tendency be described in a word? Perhaps facing up to the ugly side of this word can stir us into action to remedy what seems to be an extremely grave crisis in Western education.

.... The first step toward progress in crucial social problems is to recognize the deceptive illusions bred by seeing only the surface of issues, of seeing only a myriad of small areas to be dealt with by specialists, one for each area. Piecemeal superficiality won't work.

... Teaching is not a matter of pouring knowledge from one mind into another as one pours water from one glass into another. It is more like one candle igniting another. Each candle burns with its own fuel. The true teacher awakens a love for truth and beauty in the heart--not the mind--of a student after which the student moves forward with powerful interest under the gentle guidance of the teacher. (Isn't it interesting how the mention of these two most important goals of learning--truth and beauty--now evokes snickers and ridicule, almost as if by instinct, from those who shrink from all that is not superficial.) These kinds of teachers will inspire love of mathematics, while so many at present diffuse a distaste for it through their own ignorance and clear lack of delight in a very delightful subject.