Exploitation of Cyclostationarity for Identifying the Volterra Kernels of Nonlinear Systems

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Abstract—**A class of random time-series inputs for nonlinear time-invariant systems that enable the analytical specification of a set of operators on the input that are orthonormal over all time to the Volterra operators for all orders and all lag sets is introduced. The time-series in this class are cyclostationary and complex-valued. One member of the class of inputs renders the Volterra operators orthonormal to themselves, in which case the Volterra series is a Fourier series. The orthonormal operators are used to obtain an input-output type of crosscorrelation formula for identifying the individual Volterra kernels of arbitrary order for a nonlinear system of possibly infinite order and possibly infinite memory. The real parts of the complex-valued inputs in this class comprise a class of real-valued inputs for which the same sets of specified operators apply. However, the orthogonality for different orders holds for these real inputs only for Volterra operators of order less than the order of the specified operator. Thus, these real inputs can be used to identify Volterra kernels only for finite-order systems. Frequency-domain counterparts of the time-domain methods that can utilize an FFT algorithm also are developed.**

Index Terms—**Volterra kernels, nonlinear system identification, cyclostationarity**

I. Introduction

THE VOLTERRA SERIES is one of the most intuitive general representations for nonlinear time-invariant systems. Unlike other familiar representations, such as the Wiener series [26] and the Cameron–Martin series [8], each term in the Volterra series completely characterizes a particular order of nonlinearity in the system. That is, the nth term, which is a linear combination over the lags j_n of the nth order lag-products

$$\lambda_n(k, j_n, x(\cdot)) \triangleq \prod_{r=1}^{n} x(k - j_{n_r}) \tag{1}$$

of the input $x(k)$, characterizes the nth-order nonlinearity in the system. In this expression, k represents discrete time and takes on integer values and j_{n_r} represents the n_rth of n lag values and $j_n = [j_{n_1}, j_{n_2}, \ldots, j_{n_n}]$. Linear systems contain only the first-order term, quadratic systems contain only the first- and second-order terms, and so on. Thus, the Volterra series representation for a nonlinear time-invariant system with input $x(k)$ and output $y(k)$ is given by

$$y(k) = \sum_{n=1}^{\infty} \left[\sum_{j_n} h_n(j_n) \lambda_n(k, j_n, x(\cdot)) \right], \tag{2}$$

where the coefficient $h_n(\cdot)$ in this linear combination is the nth-order Volterra kernel.

It is assumed that the finite-term counterpart of the symbolic representation (2) converges as the number of terms indexed by n in the series approaches infinity. For example, if a system with input $x(k)$ and output $y(k)$ has the property that small changes in $x(k)$ result in only small changes in $y(k)$, and if this system has finite memory, then it follows from the Stone–Weierstrass theorem that this system can be uniformly approximated over a uniformly bounded set of inputs by a Volterra series [22]. However, it is mentioned that the assumption of convergence necessarily rules out application to systems amenable to identification by other methods, such as that of Wiener, Lee, and Schetzen [26], which are based on other series representations such as the Wiener series, that do converge in some instances where the Volterra series diverges. For continuous time, the completeness of the Volterra basis was first discussed and proved by Frechet in 1910 [9], and the sufficiency for convergence of a fading memory was first established by Wiener in 1958 [28, p. 41]. Wieners work was explained and developed further by Schetzen in 1980 [26, pp. 334, 452], Sandberg, and others.

The theory of Volterra series has been applied to practical problems in diverse fields of science and engineering, such as discrete- and continuous-time systems, electrical circuits, communication systems, fluid mechanics, rheology, hydrology, structural engineering, and physiology (cf. [6], [15], [21], [25], [26]).

Many methods that utilize crosscorrelations between the output and transformations of the input to identify the Volterra kernels are based on the use of stationary random time-series as system inputs (cf. [6], [16], [29]). This can present difficulties because the lag-product operators, or Volterra operators[1]

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[1] It is common to refer to the Volterra (and Wiener, Fourier, and other) operators as functionals. But, the range of a functional is the reals, and since we find it more appropriate here to interpret the quantities in the ranges of these transformations as functions (or vectors) in the same function space (or vector space) as the quantities in the domains of these transformations, then we are obliged to call these transformations operators. When we speak of the orthogonality of two operators we mean the orthogonality of the two functions in their range corresponding to a given function in their domain, viz., $x(k)$. Since the nth term in the Volterra series is appropriately called the

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(1) in the Volterra series are generally not orthogonal to each other,

$$\langle \lambda_n(k, \boldsymbol{k}_n, x(\cdot)\lambda_m^*(k, \boldsymbol{j}_m, x(\cdot)))\rangle \neq 0, \tag{3}$$

for $n \neq m$ or $\boldsymbol{k}_n \neq \boldsymbol{j}_n$, for stationary inputs. (In (3), $(\cdot)^*$ denotes complex conjugation and $\langle\cdot\rangle$ denotes average over the integers k.) Moreover, no explicit[2] set of operators on the input $x(k)$ that is orthonormal to the infinite set of Volterra operators is known (cf. [26], [27]). A popular approach to circumventing this problem is to use an alternative series representation that does have orthogonal operators and then, after identifying the kernels for the operators in the alternative series, to convert the identified kernels into Volterra kernels. One of the most popular such approaches, that of Wiener, Lee, and Schetzen, uses the Wiener series, which has orthogonal operators when the input $x(k)$ is a stationary white Gaussian time-series [21], [26]. Other examples are discussed in [16], [29] where the role of orthogonality is clarified. In general, for an infinite-order system, *all* kernels in the alternative series must be identified before any Volterra kernel can be determined. Consequently, the system must be accurately approximable by a finite-term series, and a number N of terms that is adequate must be known. However, for a finite-order system, there *are* operators on the input $x(k)$ that are orthogonal to all but the highest order Volterra operators. In fact, this is one of the properties of the Wiener operators for a white Gaussian input. Thus by identifying the highest-order Volterra kernel and then subtracting its effect from the system output, the same procedure can be used repeatedly to identify all Volterra kernels down to the first-order. (This is proved as a corollary to the main result in this paper.)

An alternative for finite-order systems is to specify a particular input (e.g., pseudo-random) over a particular time interval (ideally assumed to be long enough that the input can be rich enough to possess all orthogonality properties needed, but not so long as to cause an unnecessary computational burden), and to specify adequately large estimates of the system order N and system memory L, and then to compute a set of operators on this input that are orthogonal to all lag-product operators (for all orders $n \leq N$ and all L^n lag-sets for each order n) on the input that could occur in the linear combination of terms that forms the Volterra series for the output. This orthogonal set of operators can then be used one at a time to identify each of the N Volterra kernels at each of the L^n points in its domain by simply forming a single inner product of the system output with the output of the appropriate orthogonal operator on the input. Since the orthogonalization of a potentially immense set of operators (more than L^N) can be computationally prohibitive, research is currently being pursued in an attempt to develop efficient methods for adaptively determining a parsimonious subset of the orthogonal operators [17], [18].

I, the lag-products in this term are more appropriately called the *elementary Volterra operators*. These two operators are identical to each other only when the Volterra kernel is unity for one lag set j_n and zero for all others. However, for economy of terminology, the modifier *elementary* is omitted in this paper.

[2]Some such sets of operators are known implicitly. For example, if $x(k)$ is a white Gaussian time-series, then each operator in the desired set can be expressed as an infinite series of scaled Wiener operators [26].

Another alternative is to directly identify the multi-dimensional Fourier transforms of kernels, such as Wiener kernels or Volterra kernels, using measurements of higher-order cross spectra. Some computational savings can be obtained in this way (e.g., [2], [10]).

In this paper, we consider a new approach. We introduce a class of random time-series inputs for nonlinear time-invariant systems that enable the analytical specification of a set of operators on the input that are orthonormal over all time to the Volterra operators for all orders and all lag sets. The time-series in this class are cyclostationary [11], [14] rather than stationary and are complex-valued. One member of the class of inputs renders the Volterra operators orthonormal to themselves, in which case the Volterra series is a Fourier series. The orthonormal operators are used to obtain an input–output type of crosscorrelation formula for identifying the individual Volterra kernels of arbitrary order for a nonlinear system of possibly infinite order and possibly infinite memory. The real parts of the complex-valued inputs in this class comprise a class of real-valued inputs for which the same sets of specified operators apply. However, the orthogonality for different orders for these real inputs holds only for Volterra operators of order less than the order of the specified operator. Thus, these real inputs can be used to identify Volterra kernels only for finite-order systems.

Two specific examples of inputs in each of the two classes are used to obtain explicit closed forms for the general identification formula. One of these examples in each of the two classes includes the option of a finite-state input. The close relationship of the other of the two methods in each of the classes to the classical Wiener–Lee–Schetzen crosscorrelation method for identifying the Wiener (and Volterra) kernels of a nonlinear system with white Gaussian input is explained.

To our knowledge, the methods introduced here for complex inputs are the first methods to be able to identify an arbitrary-order Volterra kernel for an infinite-order system. Unfortunately, these methods can be used only when a complete description of the input–output rule (such as a mathematical model) that defines the system for complex inputs is available, e.g., for simulation on a computer. Complex inputs cannot, of course, be applied to physical systems. Nevertheless, methods for identification of the Volterra kernels from mathematical models have important practical applications, such as the calculation of the performance of a communication system with nonlinear components (cf. [3], [5]). Thus, the methods introduced here can be viable alternatives to analytical methods that can be applied when a mathematical model of the system is available (cf. [26]).

Although the methods for the real counterparts of the complex inputs require that the system be accurately approximable by a finite-term Volterra series with known number of terms, the new finite-state methods require considerably less correlation time for a given level of statistical reliability than the method of Wiener–Lee–Schetzen. Furthermore, unlike previously considered finite-state inputs (cf. [20], [24]), the range of input values is a continuum rather than a finite set of amplitudes. These finite-state inputs therefore appear to be richer than the finite-amplitude inputs [24], [27].

The approach to nonlinear system identification taken in this paper was originally motivated by the theory of cyclostationary time-series. The four specific methods given as examples of the general method were originally derived in terms of cyclic cross moments between the input and output of the system in a step-by-step manner. That is, a formula for identifying the n-th order Volterra kernel from cyclic cross moment measurements was derived first for $n = 1$, then $n = 2$, then $n = 3$, and so on until the general solution for arbitrary n was evident [1]. Later, with insight gained from Yasuis interpretation [16], [29] of the Wiener–Lee–Schetzen and other crosscorrelation-based methods in terms of orthogonality, the concise, general, and more transparent derivation presented in this paper was developed.

The simplicity of the crosscorrelation formulas for identification of the time-domain Volterra kernels can be exploited to obtain by a straightforward method frequency-domain counterparts for direct identification of the *Volterra transfer functions*—the multidimensional Fourier transforms of the Volterra kernels. This also is accomplished in this paper.

To put the new methods in perspective, it should be mentioned that there are quite a number of approaches to identification of Volterra kernels and Volterra transfer functions that are not based directly on crosscorrelating the system output with transformations of a random input. These include methods based on transient inputs (e.g., sums of shifted impulses) and periodic inputs (sums of sinewaves or, in the frequency domain, sums of shifted impulses) (cf. [3], [26]). There also are the structurally constrained approaches that assume the nonlinear system has a particular structure typically consisting of cascade and/or parallel connections of linear subsystems with memory and nonlinear subsystems without memory (cf. [4], [7], [19]), and then identify the subsystems in the assumed structure. The Volterra kernels can be computed directly from the identified subsystems.

II. The General Method

We shall use the framework of temporally random discrete-time-series [11], [14], rather than the more commonly used framework of stochastic processes [12], because the former relates more directly to the way system identification methods are implemented in practice (cf. [27]), whereas the latter introduces irrelevant abstractions. We need only a few basic results from the theory of temporally random time-series.

A *stationary* time-series $x(k)$ is one for which infinite-time averages of lag-products (and all other well-behaved time-invariantor k-invariantfunctions of the time-series) exist and are not all identically zero. A *purely stationary* time-series is a stationary time-series for which the sinusoidally weighted lag-products (and all other well-behaved time-invariant functions) have infinite-time averages that are identically zero for all noninteger sinewave frequencies [11], [13]. That is, the cyclic moments

$$R_x^\alpha(\boldsymbol{j}_n) \neq \left\langle \lambda_n(k, \boldsymbol{j}_n, x(\cdot)) e^{-i2\pi\alpha k} \right\rangle \tag{4}$$

are zero for all noninteger real cycle frequencies α, all lag variables j_n, and all positive integers n. It follows that if $z(k)$ is purely stationary, then every k-invariant function of $z(k - k_1), z(k - k_2), \cdots, z(k - k_n)$, say

$$\begin{aligned} f(k, \boldsymbol{k}_n, z(\cdot)) &= f(z(k - k_1), \cdots, z(k - k_n)) \\ &= f(0, \boldsymbol{k}_n - \boldsymbol{1}_n k, z(\cdot)) \end{aligned}$$

(where 1_n is the n-dimensional column vector with all elements equal to unity) is purely stationary, e.g.,

$$\left\langle f(k, \boldsymbol{k}_n, z(\cdot)) e^{-i2\pi\alpha k} \right\rangle = 0, \tag{5}$$

for noninteger α. If (4) is nonzero for some noninteger α, then $x(k)$ is *cyclostationary* (or *exhibits cyclostationarity*) with cycle frequency (or frequencies) a [11], [13].

If the joint fraction-of-time amplitude densities [11]–[13] of a time-series factor into products of the individual fraction-of-time densities, then the time-series is said to be *white* (in the strict sense); i.e., it is a sequence of statistically (temporally) independent variables. It follows that if $z(k)$ is a white time-series, then its joint moments factor as follows:

$$\left\langle \prod_{q=1}^{p} z^{n_q}(k - j_{m_q}) \right\rangle = \prod_{q=1}^{p} \left\langle z^{n_q}(k - j_{m_q}) \right\rangle, \tag{6}$$

for distinct lags $j_{m_1}, j_{m_2}, \cdots, j_{m_p}$. We shall consider first the class of cyclostationary inputs of the form

$$x(k) = z(k)s(k), \tag{7}$$

where $z(k)$ is a real-valued or complex-valued purely stationary time-series (not necessarily white although $z(k)$ is white in all specific examples presented), and $s(k)$ is the trivial cyclostationary time-series

$$s(k) = e^{i\omega k}, \tag{8}$$

for some real irrational $\omega/2\pi$. (Rational $\omega/2\pi$ are problematic because for some nonzero integers n, $e^{i\omega k}$ and $(e^{i\omega k})^n$ are not orthogonal.) Then, we shall consider real inputs of the form

$$x(k) = \mathrm{Re}\{z(k)s(k)\}. \tag{9}$$

We shall see that for some purely stationary time-series $z(k)$ we can find a set of k-invariant operators $\phi_n(k, \boldsymbol{k}_n, z(\cdot))$ for a given $z(k)$ that is orthonormal (reciprocal) to the set of lag-product operators

$$\lambda_n(k, \boldsymbol{j}_n, z(\cdot)) = \prod_{r=1}^{n} z(k - j_{n_r}) \tag{10}$$

in the sense that

$$\langle \lambda_n(k, \boldsymbol{j}_n, z(\cdot)) \varphi_n^*(k, \boldsymbol{k}_n, z(\cdot)) \rangle = \delta_{\tilde{\boldsymbol{j}}_n \tilde{\boldsymbol{k}}_n}, \tag{11}$$

where $\tilde{\boldsymbol{j}}_n$ is any permutation of $\boldsymbol{j}_n$, and is the n-dimensional Kronecker delta (which is the product of n one-dimensional Kronecker deltas: $\delta_{j_{n_r} k_{n_s}} = 1$ for $j_{n_r} = k_{n_s}$ and $\delta_{j_{n_r} k_{n_s}} = 0$ for $j_{n_r} \neq k_{n_s}$).

The preceding concepts and definitions in this section are all that are needed to develop methods for identifying the symmetrized kernels $h_n(j_n)$ in the Volterra series representation (2) for the output $y(k)$ of a nonlinear time-invariant

system with input $x(k)$. (The kernels of any system can be symmetrized by averaging $h_n(j_n)$ over all permutations $\tilde{j}_n$ without affecting the output of the system [26].)

Theorem 1: Let the k-invariant operators γ_n be defined by

$$\gamma_n(k, \boldsymbol{k}_n, z(\cdot)) \triangleq e^{-i\omega \mathbf{1}_n'(\boldsymbol{k}_n - \mathbf{1}_n k)} \varphi_n(k, \boldsymbol{k}_n, z(\cdot)), \quad (12)$$

where φ_n satisfies (11), $\mathbf{1}_n'$ denotes the transpose of the column vector $\mathbf{1}_n$, and let $x(k)$ be given by (7) (8). Then these operators γ_n are orthonormal over both the orders n and lag sets $\boldsymbol{k}_n$ to the Volterra operators λ_m :

$$\langle \lambda_m(k, \boldsymbol{j}_m, x(\cdot)) \gamma_n^*(k, \boldsymbol{k}_n, z(\cdot)) \rangle = \delta_{nm} \delta_{\tilde{\boldsymbol{j}}_n \boldsymbol{k}_n}. \quad (13)$$

A proof of this theorem is given in the Appendix. As a corollary, we have the following result on identifying Volterra kernels.

Corollary: Given operators φ_n satisfying (11) and system input specified by (7)–(8), the operators γ_n defined by (12) can be used to identify the Volterra kernels in (2) by performing the crosscorrelation operation

$$h_n(\boldsymbol{k}_n) = \frac{1}{P(\boldsymbol{k}_n)} \langle y(k) \gamma_n^*(k, \boldsymbol{k}_n, z(\cdot)) \rangle, \quad (14)$$

where $P(\boldsymbol{k}_n)$ is the number of distinct permutations of the elements of $\boldsymbol{k}_n$. Specifically, if $\boldsymbol{k}_n = [k_{m_1}, \cdots, k_{m_1}, k_{m_2}, \cdots, k_{m_2}, \cdots, k_{m_p}, \cdots, k_{m_p}]$ where k_{m_q} is repeated n_q times and $n_1 + n_2 + + n_p = n$, then

$$P(\boldsymbol{k}_n) = \frac{n!}{n_1! n_2! \cdots n_p!}. \quad (15)$$

For the real inputs (9), we have

$$x(k) = \frac{1}{2} z(k) e^{i\omega k} + \frac{1}{2} z^*(k) e^{-i\omega k} \quad (16)$$

and, parallel to Theorem 1, we have the following theorem (which is proved in the Appendix).

Theorem 2: Let the operators γ_n be defined by (12) modified to include the factor 2^n, where φ_n satisfies (11), and let $x(k)$ be given by (16). Then these operators γ_n are orthonormal over both the orders n and lag sets $\boldsymbol{k}_n$ to the Volterra operators λ_n of orders $m \leq n$:

$$\langle \lambda_m(k, \boldsymbol{j}_m, x(\cdot)) \gamma_n^*(k, \boldsymbol{k}_n, z(\cdot)) \rangle = \delta_{nm} \delta_{\tilde{\boldsymbol{j}}_n \boldsymbol{k}_n}. \quad (17)$$

Thus, parallel to the corollary, for an Nth-order Volterra system (for which $h_m \equiv 0$ for all $m > N$ in (2)), (14) still holds for $n = N$. After (14) is used to identify the Nth-order kernel, the Nth term in (2) can be subtracted off to produce the output of an $(N-1)$th-order system whose highest order kernel $h_{N-1}(j_{N-1})$ can then be identified. (If we use this procedure, then we no longer require the sinusoidal factor $s(k)$ in (7) (i.e., we can simply use $x(k) = z(k)$ for real $z(k)$ in (16)) because (13) can be satisfied for $m \leq n$ with $\omega = 0$ in (12). This is established next. Nevertheless, we shall see that for finite-state inputs, the sinusoidal factor $s(k)$ results in a potential advantage.)

So, the only part of the identification scheme that is not explicitly defined is the method for obtaining the reciprocal set of operators $\phi_n(k, \boldsymbol{k}_n, z(\cdot))$ which satisfy (11) for a given purely stationary time-series $z(\cdot)$. We shall accomplish this for the special case of white $z(\cdot)$.

Lemma: Let $z(k)$ be a purely stationary white time-series for which the powers $z^m(k)$ are linearly independent. In general, there exists a set of univariate orthogonal operators $\psi_n(z(k - k_o))$ that are orthonormal to the particular lag-product operators $\lambda_n(k, j_m, z(\cdot)) = z^m(k - j_o)$ for $m \leq n$,

$$\langle z^m(k - j_o) \psi_n^*(z(k - k_o)) \rangle = \delta_{mn} \delta_{j_o k_o}, \qquad \text{for } m \leq n, \quad (18)$$

and we can define operators to satisfy the orthonormality property (11) and, for $\omega = 0$ in (12), to also satisfy the orthonormality property (13) for $m \leq n$, as follows:

$$\varphi_n(k, \boldsymbol{k}_n, z(\cdot)) \triangleq \prod_{q=1}^{p} \psi_{n_q}(z(k - k_{m_q})), \quad (19)$$

where n_q is the number of times the lag k_{m_q} is repeated.

This Lemma is proved in the Appendix.

III. Examples and Discussion

Example 1 (PM): Let $z(k) = \sigma e^{i\theta(k)}$ be a purely stationary sequence of statistically independent (white) variables having an M-ary discrete uniform circular temporal-probability distribution (fraction-of-time distribution [11]–[13]), which results from $\theta(k)$ having an M-ary discrete uniform distribution in the interval $[-\pi, \pi]$. In this case, we can make the choice

$$\psi_n(z(k - r)) \triangleq \sigma^{-n} z^n(k - r) \quad (20)$$

and satisfy (18) provided that nm is not a nonzero integer multiple of M. Therefore, a sufficient condition for identification formula (14) to be valid is that n_q in (19), which is used in (14), be less than M. That is, the alphabet size M must exceed the number of times any lag value is repeated at the point in the domain of the kernel at which the kernel is to be identified. (This restriction is a result of the fact that $z^m(k)$ are linearly independent only for $1 \leq m \leq M$.) This can be guaranteed for arbitrarily high-order kernels, by letting $M \to \infty$; i.e., by using a continuous uniform distribution for $\theta(k)$.

It follows from (20) that for the complex PM input, we have

$$\psi_n(k, \boldsymbol{k}_n, z(\cdot)) = \sigma^{-n} \lambda_n(k, \boldsymbol{k}_n, z(\cdot)) \quad (21)$$

and, therefore,

$$\psi_n(k, \boldsymbol{k}_n, z(\cdot)) = \sigma^{-n} \lambda_n(k, \boldsymbol{k}_n, x(\cdot)). \quad (22)$$

Consequently, (13) reveals that lag-product operators are orthogonal for either different orders n and/or different lag sets $\boldsymbol{k}_n$. Because of this, the Volterra series of operators is a Fourier series when the system input is complex PM with an alphabet size M that exceeds the order N of the system (since N is the maximum number of times a lag value can be repeated within a given kernel). This means that the approximation obtained using only the N' identified terms with the largest time-averaged squared values, for any N' less than N, has the smallest time-averaged squared error, for this particular input, between the outputs of the actual Nth-order system and the

approximating N'th-order system, among all approximations with N' or fewer terms (assuming perfect identification of each term, which requires infinite correlation time).

The real counterpart (9) of the input in this example is the phase-modulated (PM) sinewave

$$x(k) = \sigma \cos(\omega k = \theta(k)). \tag{23}$$

It is important to recognize that for $\omega \neq 0$, both the complex and real PM inputs take on values throughout a continuum (a disk of diameter 2σ for complex PM and an interval length of 2σ for real PM) even though the random sequence $\theta(k)$ has only a finite number (M) of states for each k (the same finite set of states for all k). For $\omega = 0$, the PM inputs take on only a finite number (M) of values. Thus, for $\omega \neq 0$, the PM input is apparently richer than it is for $\omega = 0$. Specifically, it is a white input with a fraction-of-time amplitude density equal to that of a sinusoid. This PM example is generalized following Example 2.

Example 2 (AM): Let $z(k)$ be a purely stationary sequence of statistically independent real-valued Gaussian variables with standard deviation σ. In this case, the scaled (nonnormal) Hermite polynomials

$$\psi_n(z(k-r)) \triangleq \frac{\mathcal{H}_n(z(k-r))}{\left\langle |\mathcal{H}_n(z(k))|^2 \right\rangle}, \tag{24a}$$

where

$$\mathcal{H}_n(z) \triangleq \sum_{q=0}^{\lfloor n/2 \rfloor} \frac{(-1)^q n!}{2^q q!(n-2q)!} z^{n-2q} \tag{24b}$$

and

$$\left\langle |\mathcal{H}_n(z(k))|^2 \right\rangle = n! \sigma^{2n}, \tag{24c}$$

and (24c) satisfy (18) (cf. [26]).

If we let $\omega = 0$ in (8), then (13) is valid for a white Gaussian time-series $z(k)$ only if the lag-product operators λ_n are replaced with the Wiener operators. Thus, in this special case, (14) with $\omega = 0$ in (12) identifies the Wiener kernels, not the Volterra kernels, and is identical to the Wiener–Lee–Schetzen method [26]. However, for a finite-order system, the highest order Volterra kernel is identical to the highest order Wiener kernel.

The real counterpart (9) of the input in this example is the amplitude modulated (AM) sinewave

$$x(k) = z(k) \cos(\omega k). \tag{25}$$

The class of inputs (7)–(9) for which the general approach developed herein applies includes, in addition to the AM and PM examples considered, joint AM/PM with distributions for the amplitude and phase other than the two considered here. For example, if the constellation of points in the complex plane corresponding to the amplitude/phase pairs for an AM/PM time-series with finite alphabet consists of concentric circles centered at the origin, and if each circle contains a uniform distribution of equally probable points, then (20) applies and the condition $n_q < M$ discussed in Example 1 refers to the minimum number, M, of points on any one circle rather than to the overall alphabet size.

It is emphasized that the two distinct methods, complex AM and complex PM (as well as complex AM/PM), can identify any Volterra kernel in any nonlinear system that admits the Volterra series representation, including infinite order and infinite memory. However, it must also be emphasized that because these two methods use complex-valued inputs $x(k)$, they can only be implemented when a complete description of the input-output rule (such as a mathematical model) that defines the system for complex inputs is available, e.g., for simulation on a computer. Nevertheless, results in Section V suggest that the PM (and presumably AM/PM) method can be a computationally attractive alternative to other methods for determining the Volterra kernels from a mathematical model of a nonlinear system. Also, for physical systems that are accurately approximable by finite-order Volterra series, the simulation results in Section V suggest that the PM (and presumably AM/PM) method can be an attractive alternative to other methods, such as that of Wiener–Lee–Schetzen that uses Gaussian inputs, since the PM method apparently requires less averaging time to obtain statistical reliability. Nevertheless, these apparent advantages have been seen in only the relatively small set of simulations that have been done so far. More work is needed to establish more firmly the advantages (and disadvantages) of these new methods relative to other methods.

IV. Frequency-Domain Counterparts

Frequency-domain counterparts of the time-domain methods, which directly identify the multidimensional Fourier transforms of the Volterra kernels (the Volterra transfer functions) and provide for the use of computationally efficient FFT algorithms in place of the many correlation operations required by the time-domain methods, can be derived as follows. We first take the n-dimensional Fourier transform of the nth-order kernel over a hypercube domain

$$\mathcal{H}_n(\boldsymbol{f}_n) = \sum_{0 \leq \boldsymbol{k}_n \leq \boldsymbol{K}} h_n(\boldsymbol{k}_n) e^{-i2\pi \boldsymbol{f}_n' \boldsymbol{k}_n}, \tag{26}$$

where $0 \leq \boldsymbol{k}_n \leq K$ means $0 \leq k_{n_i} \leq K$ for $i = 1, 2, \cdots, n$, and where K should be chosen to exceed the system memory length L (but not excessively), for which $h_n(\boldsymbol{k}_n) = 0$ for $k_{n_i} > L$ for any i. Then, we substitute the kernel estimate (14), with the averaging time for $\langle \cdot \rangle$ reduced from inf to T, into (26) to obtain the estimate

$$\hat{H}_n(\boldsymbol{f}_n) = \sum_{0 \leq \boldsymbol{k}_n \leq \boldsymbol{K}} \left\langle y(k) \frac{1}{P(\boldsymbol{k}_n)} \gamma_n^*(k, \boldsymbol{k}_n, z(\cdot)) \right\rangle_T e^{-i2\pi \boldsymbol{f}_n' \boldsymbol{k}_n}, \tag{27}$$

Given that γ_n is k-invariant,

$$\gamma_n^*(k, \boldsymbol{k}_n, z(\cdot)) = \gamma_n^*(0, \boldsymbol{k}_n - 1 - k\mathbf{1}_n, z(\cdot)), \tag{28}$$

and observing that

$$P(\boldsymbol{k}_n - k\mathbf{1}_n) = P(\boldsymbol{k}_n), \tag{29}$$

we can reexpress (27) as

$$\hat{H}_n(\boldsymbol{f}_n) = \sum_{0 \le \boldsymbol{k}_n \le \boldsymbol{K}} \langle y(k) w^*(\boldsymbol{k}_n - k\mathbf{1}_n) \rangle_T e^{-i2\pi \boldsymbol{f}_n' \boldsymbol{k}_n}, \quad (30)$$

where

$$w(\boldsymbol{k}_n) = \frac{1}{P(\boldsymbol{k}_n)} \gamma_n(0, \boldsymbol{k}_n, z(\cdot)). \quad (31)$$

Now, by truncating each of $y(k)$ and $w(\boldsymbol{k}_n - k\mathbf{1}_n)$ to intervals of length T, and denoting these truncated versions by $y_T(\cdot)$ and $w_T(\cdot)$, we can closely approximate (30) for $T \gg K$ by

$$\begin{aligned} &\hat{H}_n(\boldsymbol{f}_n) \\ &\cong \sum_{0 \le \boldsymbol{k}_n \le \boldsymbol{K}} \left[\frac{1}{T} \sum_{k=-\infty}^{\infty} y_T(k) w_T^*(\boldsymbol{k}_n - k\mathbf{1}_n) \right] e^{-i2\pi \boldsymbol{f}_n' \boldsymbol{k}_n}, \\ &= \left[\frac{1}{T} W_T^*(-\boldsymbol{f}_n) Y_T(\boldsymbol{f}_n' \mathbf{1}_n) \right] \otimes A_K(f_1) \otimes \cdots \otimes A_K(f_n), \end{aligned} \quad (32)$$

where

$$A_K(f) = \sum_{k=-K/2}^{K/2} e^{-i\pi f k} = \frac{\sin[\pi f(K+1)]}{\sin(\pi f)}, \quad (33)$$

$$Y_T(f) = \sum_{|k| \le T/2} y(k) e^{-i\pi f k}, \quad (34)$$

$$W_T(-\boldsymbol{f}_n) = \sum_{|k_n| \le T/2} w(\boldsymbol{k}_n) e^{i2\pi \boldsymbol{f}_n' \boldsymbol{k}_n}. \quad (35)$$

The equality (32) is a direct result of the convolution theorem for the Fourier transform.

For fixed K, in the limit as $T \to \infty$, the approximation (32) becomes exact and, because of the corollary to Theorem 1, the limit of (32) equals the Volterra transfer function of the system with finite memory length $L < K$. If the system has infinite memory length, but its Volterra transfer functions exist (the series (26) converges as $K \to \infty$), then the limit of (32) as $K \to \infty$ equals the Volterra transfer function of the system.

In summary, given the set of operators γ_n defined by (12) and (19), which are orthonormal to the Volterra operators, we can use an FFT algorithm to compute (34) and (using (31)) (35); then we can multiply these transforms and perform an n-dimensional convolution with the spectral window $A(f)$ to obtain the Volterra transfer function estimate (32). The computational efficiency of the n-dimensional convolution is greatly enhanced by replacing the window (33) with a rectangular window of width on the order of $1/K$. This can also improve the fidelity of the estimate (32).

Because of the factorization (19) of the factor γ_n in $w(\boldsymbol{k}_n)$ (cf. (31)), the n-dimensional Fourier transform of this factor is simply the product of $p \le n$ one-dimensional transforms. Thus, the transform (35) can be obtained (from the convolution theorem) by convolving the transforms of the two factors γ_n and $1/P(\boldsymbol{k}_n)$ in (31). The transform of $1/P(\boldsymbol{k}_n)$ can be computed in advance of experimentation since it does not depend on the data $x(k)$.

The additional factorizations (20) and (24) of $\psi_n(z)$ into products of z in (20) and a sum of weighted products of z in (24) reveal that for both the PM and AM methods, the FFT's of the factor γ_n in $w(\boldsymbol{k}_n)$ can all be obtained from a few FFT's of the data $z(k)$ (or shifted FFT's of $x(k)$) and some of its powers (e.g., powers n_q from (19) for (20)).

TABLE I
BIAS AND COEFFICIENT OF VARIATION FOR VOLTERRA KERNEL ESTIMATES OBTAINED USING VARIOUS METHODS

Method	Record Length = 1000		Record Length = 10 000	
	|bias|	coef. var	|bias|	coef. var
CPM $\omega = 0$	0.2%	0.002	0.07%	0.0002
CPM $\omega = 1$	0.3%	0.002	0.05%	0.0002
RPM $\omega = 0$	1%	0.01	0.4%	0.001
RPM $\omega = 1$	0.8%	0.008	0.2%	0.0008
CAM $\omega = 1$	2.4%	0.03	0.3%	0.003
RAM $\omega = 0$	0.4%	0.03	0.4%	0.004

V. SIMULATIONS

To verify that the new system identification methods do indeed provide estimates of Volterra kernels that converge to the correct values, and to obtain a preliminary indication of convergence time for a specific system, we simulated a second-order Volterra system consisting of a linear time-invariant transformation described by the first-order difference equation

$$-0.89897z(k-1) + 1.89897z(k) = x(k),$$

followed by a squaring operation

$$y(k) = z^2(k).$$

System inputs of both the AM and PM types were simulated with sine-wave frequencies of $\omega = 0$ and 1 and, for PM, alphabet sizes ranging from $M = 3$ to $M = 12$. All four methods corresponding to real and complex versions of AM and PM, as described in Examples 1 and 2 in Section III, were tested. Single-precision input-data records, and algorithms (including a uniform noise generator, and Gaussian noise generator taken from [23]) coded in FORTRAN on a SUN SPARCstation II were used for the simulations. One hundred Monte Carlo trials were used for each case, input-data record lengths of 1000 and 10 000 were used for each method, and the bias and coefficient of variation (variance divided by squared true-value) for the particular point $(k_1 = 0, k_2 = 1)$ in the domain of the second-order kernel $h_2(k_1, k_2)$ were computed. The abbreviations RAM, CAM, RPM and CPM are used in Table I for the real AM, complex AM, real PM, and complex PM methods, respectively. (Note, however, that RAM = CAM for $\omega = 0$.) Since the variations in bias and coefficient of variation with respect to the alphabet size M for PM were considered to be statistically insignificant (for the small ensemble size of 100), only the average values of these parameters (over the set of alphabet sizes) are recorded in Table I.

It can be seen from the results in Table I that both bias and coefficient of variation are quite small and decrease as the record length over which time averaging is carried out is increased. For a ten-fold increase in averaging time, there is a ten-fold decrease in coefficient of variation. The reduction in bias, on the other hand, varies more from one method to another. For the relatively small number of trials (100), the statistical reliability of the values for the AM methods (especially for the shorter record length of 1000) are suspect.[3] Nevertheless, the approximate ten-fold (five-fold) reduction in coefficients of variation of the CPM (RPM) method over the AM methods for record lengths of 10 000 is significant and suggests that these finite-state-input methods converge substantially more rapidly than the Gaussian-input methods which include the Wiener–Lee–Schetzen method (RAM with $\omega = 0$). Similar results were obtained in simulations of a third-order Volterra system performed with different software on an HP 9000-835 computer.

VI. Conclusion

A new approach to obtaining sets of operators that are orthogonal to the Volterra operators for a class of input time-series is described. It is shown how these orthogonal operators can be used in a crosscorrelation-based method to identify the Volterra kernels of nonlinear time-invariant systems. This class of inputs includes both Gaussian and finite-state time-series, and simulations suggest that methods used with the finite-state inputs converge considerably more rapidly than the methods used with Gaussian inputs, which includes as a special case the method of Wiener, Lee, and Schetzen.

For complex-valued inputs, the new methods are believed to be the only crosscorrelation-based methods known that identify arbitrary order Volterra kernels of an infinite-order system (with infinite memory). Such methods can be used to compute the Volterra kernels of nonlinear systems that can be simulated on a computer.

The computational efficiency of these new crosscorrelation-based methods can be increased by using an FFT algorithm to implement the frequency-domain counterparts developed herein that are based on frequency-smoothed cross-spectra.

VII. Appendix
Proof of Theorem 1, Theorem 2, and Lemma

Proof of Theorem 1: Substituting (7) and (8) into (1) and the result together with (12) into the left-hand side of (13) yields

$$\begin{aligned}(13) &= \Big\langle \lambda_m(k, \boldsymbol{j}_m, z(\cdot)) e^{i\omega\left(mk - \mathbf{1}_m'\boldsymbol{j}_m\right)} \\ &\qquad e^{i\omega\left(\mathbf{1}_n'\boldsymbol{k}_n - nk\right)} \varphi_n^*(k, \boldsymbol{k}_n, z(\cdot))\Big\rangle \\ &= \Big\langle \lambda_m(k, \boldsymbol{j}_m, z(\cdot)) \varphi_n^*(k, \boldsymbol{k}_n, z(\cdot)) \\ &\qquad e^{i\omega(m-n)k}\Big\rangle e^{i\omega\left(\mathbf{1}_n'\boldsymbol{k}_n - \mathbf{1}_m'\boldsymbol{j}_m\right)}. \end{aligned} \tag{A.1}$$

[3] Other cases ($\omega \neq 0$) for the RAM method that were tested resulted in biases as large as 25%. No explanation for this can be found.

Since $\lambda_n(k, \boldsymbol{j}_m, z(\cdot))$ and $\varphi_n^*(k, \boldsymbol{k}_n, z(\cdot))$ are k-invariant functions of $[z(k-j_1), \cdots, z(k-j_m)]$ and $[z(k-k_1), \cdots, z(k-k_n)]$, respectively, and $z(k)$ is purely stationary, then

$$\lambda_m(k, \boldsymbol{j}_m, z(\cdot))\varphi_n^*(k,)$$

is purely stationary. Consequently, the average (A.1) is zero for $m - n \neq 0$ since $\omega/2\pi$ is irrational (cf. (5)). For $m = n$, we have

$$\langle \lambda_n(k, \boldsymbol{j}_n, z(\cdot))\varphi_n^*(k, \boldsymbol{k}_n, z(\cdot))\rangle e^{i\omega\mathbf{1}_n'(\boldsymbol{k}_n - \boldsymbol{j}_n)}, \tag{A.2}$$

and (11) guarantees that this average is zero for $\tilde{\boldsymbol{j}}_n \neq \boldsymbol{k}_n$ and is unity for $\tilde{\boldsymbol{j}}_n = \boldsymbol{k}_n$. Thus, we have verified (13). □

Proof of Theorem 2: Substituting (16) into (1) and the result together with (12) (modified to include the factor 2_n) into the left-hand side of (17) yields the same result as (A.1) except that the factor

$$\lambda_m(k, \boldsymbol{j}_m, z(\cdot))e^{i\omega\left(mk - \mathbf{1}_n'\boldsymbol{j}_m\right)}$$

is replaced with

$$\sum_{\substack{p+q=m \\ 0\le p,q\le m}} \lambda_p(k, \boldsymbol{j}_p, z(\cdot))\lambda_q^*(k, \boldsymbol{j}_q, z(\cdot))e^{i\omega\left([p-q]k + \mathbf{1}_q'\boldsymbol{j}_q - \mathbf{1}_p'\boldsymbol{j}_p\right)}.$$

Using the same reasoning as in the proof of Theorem 1, we obtain the desired result (the right-hand side of (13)) provided that $p - q - n = 0$ only if $p = n$ and $q = 0$. The only way to guarantee this is to impose the restriction $m \le n$. □

Proof of Lemma: The set of operators $\psi_n(z(k))$ satisfying (18) for $j_o = k_o$ can be obtained simply by applying the Gram–Schmidt orthonormalization procedure to $z^m(k)$ (cf. [16]). For $j_o \neq k_o$, (18) is then satisfied since $z(k)$ is white and, therefore, $\psi_n(z(k))$ also is white.

Substituting (19) and the lag-product definition of $\lambda_n(k, \boldsymbol{j}_n, z(\cdot))$ into the left-hand side of (11) yields

$$(11) = \left\langle \prod_{v=1}^{n} z(k - j_{n_v}) \prod_{q=1}^{p} \psi_{n_q}^*\left(z\left(k - k_{m_q}\right)\right)\right\rangle. \tag{A.3}$$

Now, let us express the lag product of z in terms of its distinct lags $\boldsymbol{j}_n = [j_{u_1}, \cdots, j_{u_1}, j_{u_2}, \cdots, j_{u_2}, j_{u_r}, \cdots, j_{u_r}]$ where j_{u_s} is repeated ts times and $t_1 + t_2 + \cdots + t_r = n$. Then, we have

$$(11) = \left\langle \prod_{s=1}^{r} z^{t_s}(k - j_{u_s}) \prod_{q=1}^{p} \psi_{n_q}^*\left(z\left(k - k_{m_q}\right)\right)\right\rangle. \tag{A.4}$$

If $\tilde{\boldsymbol{j}}_n = \boldsymbol{k}_n$, then all the j_{u_s} and k_{m_q} match up and (6) and (18) used in (A.4) reveal that (11) = 1. But if $\tilde{\boldsymbol{j}}_n \neq \boldsymbol{k}_n$ and there is some j_{u_s} that does not equal any of the k_{m_q}, then (6) and (18) used in (A.4) reveal that (11) = 0. On the other hand, if every j_{u_s} equals some k_{m_q}, then the assumption $\tilde{\boldsymbol{j}}_n \neq \boldsymbol{k}_n$ requires that $\{t_s\}$ and $\{n_q\}$ not match up. (That is, the numbers of repetitions of the k_{m_q} do not coincide with the numbers of repetitions of j_{u_s}.) In this case, at least one of the n_q will exceed at least one of the t_s, in which case the corresponding factor in (A.4) will be zero, because of (6) and (18). Consequently, (11) must be zero if $\tilde{\boldsymbol{j}}_n \neq \boldsymbol{k}_n$. Observe that this same argument proves that the right-hand side of

(A.3) is zero if n is replaced with $m < n$. This verifies the orthogonality

$$\langle \lambda_m(j, j_M, z(\cdot))\varphi_n^*(k, \boldsymbol{k}_n, z(\cdot))\rangle = \delta_{mn}\delta_{\vec{j}_n \boldsymbol{k}_n}, \qquad m \le n,$$

which is (13) with $\omega = 0$ in (7)–(8) and (12). □

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