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Fraction-of-time probability for time-series that exhibit cyclostationarity*

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Abstract. A nonstochastic alternative to the stochastic process framework for conceptualizing, modeling and analyzing time-series encountered in communications, radar and telemetry systems is proposed. Wold's isomorphism between a single time-series and an ergodic stationary stochastic process is generalized to accommodate time-series with periodic structure and corresponding cycloergodic cyclostationary stochastic processes. This reveals the existence of a nonstochastic theory for single time-series with periodic structure that completely parallels the theory of cycloergodic cyclostationary stochastic processes. In particular, the concept of a nonstochastic stationary fraction-of-time probability (temporal-probability) model for a single time-series, which is closely associated with Wold's isomorphism, is generalized to cyclostationary and almost cyclostationary nonstochastic temporal-probability models for time-series with periodic structure corresponding to a single period and to multiple incommensurate periods, respectively. Gaussian time-series are considered as a specific illustrative case. Applications to signal processing are cited.

Zusammenfassung. Eine nichtstochastische Alternative zur Beschreibungsweise mit stochastischen Prozessen wird vorgeschlagen, wie sie für die Konzeptualisierung, Modellierung und Analyse von Zeitreihen dient, die man in Nachrichten, Radar- und Telemetriesystemen vorfindet. Der Woldsche Isomorphismus zwischen einer einzelnen Zeitreihe und einem ergodischen stationären Prozess wird verallgemeinert, sodaß er Zeitreihen mit periodischer Struktur und die entsprechenden zykloergodischen zyklostationären stochastischen Prozesse umfaßt. Dadurch wird die Existenz einer nichtstochastischen Theorie für einzelne Zeitreihen mit periodischer Struktur aufgezeigt, die der Theorie zykloergodischer zyklostationärer stochastischer Prozesse vollkommen parallel ist. Insbesondere wird das Konzept eines Modells für eine einzelne Zeitreihe, das auf einer nichtstochastischen stationären fraction-of-time Wahrscheinlichkeit (zeitliche Wahrscheinlichkeit) beruht und eng verknüpft mit dem Woldschen Isomorphismus ist, auf zyklostationäre beziehungsweise fast zyklostationäre nichtstochastische zeitliche Wahrscheinlichkeitsmodelle für Zeitreihen verallgemeinert, deren periodische Struktur eine einzige Periode beziehungsweise mehrere inkommensurable Perioden aufweist. Zur Illustrierung werden Gaußsche Zeitreihen als ein spezifischer Fall betrachtet. Anwendungen in der Signalverarbeitung werden erwähnt.

Résumé. Nous proposons une alternative non stochastique à la construction basée sur les processus stochastiques utilisée pour conceptualiser, modéliser et analyser les séries temporelles rencontrées dans les systèmes de communication, de radar et de télémétrie. L'isomorphisme de Wold entre une séries temporelle isolée et un processus stochastique stationnaire ergodique est généralisée afin de prendre en compte les séries temporelles à structure périodique et les processus stochastiques cyclo-ergodiques et cyclo-stationnaires correspondants. Ceci révèle l'existence d'une théorie non stochastique pour les séries temporelles isolées à structure périodique qui établit un parallèle complet avec la théorie des processus stochastiques cyclo-ergodiques cyclo-stationnaires. En particulier, le concept de modèle de probabilité de type 'fraction de tempts' (probabilité temporelle) non stochastique stationnaire pour une série temporelle isolée, qui est étroitement associé à l'isomorphisme de Wold, est généralisé aux modèles de probabilité temporelle cyclo-stationnaires et presque cyclo-stationnaires

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pour des séries temporelles ayant une structure périodique correspondant respectivement à une période unique et à des périodes multiples incommensurables. Nous considérons à des fins d'illustration spécifique des séries temporelles gaussiennes. Nous citons des applications au traitement du signal.

Keywords. Cyclostationary time-series, temporal probability.

1. Introduction

Wold's isomorphism [31] (developed here in Section 3) between a single persistent time-series—an ongoing function of time—and an ergodic stationary stochastic process—a time-indexed family of random variables with joint probability distributions that are invariant to time translation—is of fundamental conceptual importance in justifying the use of theoretical probabilistic models for some empirical time-series and the use of empirical time-average interpretations of theoretical expected values for some stochastic processes. It provides the basis for bringing together the two otherwise disparate philosophies that give rise to the functional models of time-series, in which statistics are obtained from finite time averages, and the stochastic models, in which statistics are obtained from finite ensemble averages (and also, in some cases, finite time averages) [18].

The ergodic theorem (which concerns the convergence of time averages of random variables) and the law of large numbers (which concerns the convergence of ensemble averages of random variables) enable us to say that in the limit, as the number of time-samples (with fixed spacing) in the time average of a given measurement on a single time-series from an ergodic stationary process approaches infinity, and in the limit, as the number of random samples in the ensemble average of the same measurement approaches infinity, the time average and ensemble average are both equal to the same quantity, the expected value. However, this equality is true only with probability equal to unity. That is, this equality *requires* that we conceive of the time-series as one sample path of the stochastic process. Thus, with this approach we are conceptually locked into the abstract probabilistic framework of stochastic processes. Nevertheless, Wold's isomorphism affords us more flexibility of thought since it establishes that although there is an isomorphism between the time-series and the stochastic process that leads to completely dual theories, we are not forced to think of the time-series as one of many possible sample paths of a stochastic process. This can be important when the mechanics of the theory of stochastic processes is useful, but the concept of a stochastic process is inappropriate because it is physically inappropriate to conceive of an ensemble of time-series. A fairly complete development of the fraction-of-time probabilistic theory of stationary and related time-series based entirely on time averages is presented in [23], and the duality between this theory and its more popular stochastic counterpart is a major theme in [8].

Wold's isomorphism also provides some justification for the somewhat indiscriminate use of both time averages and expected values that some engineers and other practically oriented analysts use to facilitate analytical derivations of formulas for model parameters such as the autocorrelation function and the spectral density of average power (where *average* can mean time average, ensemble average, or both).

The purpose of this paper is to show how to generalize Wold's isomorphism and the non-stochastic fraction-of-time probabilistic theory to accommodate time-series with periodic structure, including both a single periodicity and multiple incommensurate periodicities. This generalization is needed, for example, in applications involving signals arising in communication, radar and telemetry systems in which random (meaning unpredictable or erratic, but not necessarily stochastic, i.e., not necessarily deriving from a probability space involving an ensemble of samples) information-bearing messages modulate periodic sine-wave carriers and periodic pulse-trains [8]. Additional periodicities can arise in such applications due to periodic multiplexing of multiple signals into one signal, and periodic spectrum-spreading and other coding operations to enhance the signal's tolerance to noise and interference and to increase utilization of transmission channel capacity.

The contribution of this paper is intended to be primarily conceptual and to be accessible to a wide audience. Consequently, mathematical technicalities are minimized and the level of rigor in derivations and demonstrations is kept relatively low. Questions of existence of mathematical quantities, such as limits, are dealt with only by giving examples. Proofs are not attempted. Although this is somewhat uncommon for papers dealing in a nonsuperficial way with concepts of ergodicity and mathematical isomorphisms for stochastic processes, it is justified by the fact that the concepts dealt with here are of considerable value to practically oriented engineers and scientists, many of whom would not appreciate a more rigorous and necessarily more technical treatment. The most complete study of nonstochastic probabilistic models for single time-series in the engineering and applied mathematics literature is the chapter by Hofstetter [23], which is devoted entirely to the subject of fraction-of-time probabilistic models of stationary and related time-series. The style adopted in this paper is much like that in [23]. Although the mathematical discussion is not rigorous, in many cases the heuristic or formal arguments that are given serve as guidelines along which rigorous proofs can be constructed.

In Section 2, an explicit version of Wold's isomorphism is described, and in Section 3 it is generalized to accommodate time-series with periodic structure corresponding to a single period. In Section 4, the stochastic framework is set aside and the concept of sine-wave-component extraction is introduced and used to obtain nonstochastic temporal probability models for time-series with periodic structure corresponding to multiple incommensurate periodicities. In Section 5, these general concepts are made specific for the case of Gaussian real- and complex-valued time-series. In Section 6 the discrete-time counterparts of the continuous-time temporal probability models are presented and the relationships between these two types of models, when the discrete time-series is obtained by time sampling a continuous time-series, are explained.

The objective of this paper can be interpreted as that of presenting the strict-sense theoretical counterpart of the wide-sense theory of nonstochastic modeling of time-series that exhibit cyclostationarity, which was first presented in [9].

2. Wold's isomorphism

An isomorphism is a distance-preserving transformation between two metric spaces. Wold [31] introduced the concept of an isomorphism[1] between a set of time-series, here obtained from a set of measurement functions applied to an underlying persistent time-series, and an analogous set of jointly ergodic stationary stochastic processes, here obtained from the same set of measurement functions applied to an underlying ergodic stationary process. The set of measurement functions considered here is the set of all finite-mean-square (where *mean* refers to time average) functions of finite sets of time translates of the underlying time-series, say $x(t)$. An example of such a time-series obtained from a measurement function is

$$y(t) = g[x(t+t_1), x(t+t_2)] \\ = x(t+t_1)x(t+t_2),$$

[1] Wold [31] worked with only discrete time-series, but we shall consider their continuous-time counterparts in this paper. More importantly, Wold established that there is a complete duality between the second-order functional theory of single stationary time-series and the second-order stochastic theory of stationary processes, and he called this duality an *isomorphism*, but he did not specify any isomorphic mapping between time-series and stochastic processes, as we shall do in this paper. He also did not point out that both the duality and the isomorphism apply for any finite order, not just second order.

provided that

$$\langle y^2(t)\rangle \triangleq \lim_{Z\leftarrow\infty} \frac{1}{Z}\int_{-Z/2}^{Z/2} y^2(t)\,\mathrm{d}t < \infty. \tag{1}$$

The metric for the set of time-series is the root-mean-square difference, i.e, the distance between $y(t)$ and $z(t)$ is

$$\langle [y(t)-z(t)]^2\rangle^{1/2}.$$

The same set of measurement functions generates the set of stochastic processes of interest here. For example, for the underlying stochastic process $X(t)$, the analog of the preceding example is

$$Y(t) = g[X(t+t_1), X(t+t_2)]$$
$$= X(t+t_1)X(t+t_2),$$

provided that

$$\mathrm{E}\{Y^2(t)\} < \infty, \tag{2}$$

where $E\{\}$ denotes probabilistic expectation. Using the fundamental theorem of expectation, the finite-mean-square condition (2) can be expressed in terms of the joint probability densities for $\{X(t+t_1), X(t+t_2), \ldots, X(t+t_M)\}$. In this example, for which $M=2$, we have

$$\mathrm{E}\{Y^2(t)\} = \iint g^2(v, w) \times f_{X(t+t_1)X(t+t_2)}(v, w)\,\mathrm{d}v\,\mathrm{d}w,$$

where $f_{X(t+t_1)X(t+t_2)}(v, w)$ is the joint probability density function for $X(t+t_1)$ and $X(t+t_2)$. This function is independent of t because of the assumed stationarity of $X(t)$. The metric for this set of stochastic processes is the root-mean-square difference (where *mean* refers to expected value), i.e., the distance between $Y(t)$ and $Z(t)$ is

$$\mathrm{E}\{[Y(t)-Z(t)]^2\}^{1/2}.$$

The isomorphic mapping between these two metric spaces maps a single time series $y(t)$ into an ensemble of random samples $\{y(t, s)\}$ (indexed by s) of a hypothetical stochastic process $Y(t)$. The mapping is simply

$$y(t, s) = y(t+s). \tag{3}$$

Assuming for the moment that $\{y(t, s)\}$ can indeed be interpreted as an ensemble of random samples, and invoking the law of large numbers for a set of random samples, we conclude that the ensemble average for the process $Y(t)$ equals its expected value (with probability equal to unity):

$$\lim_{V\to\infty} \frac{1}{V}\int_{-V/2}^{V/2} y^2(t, s)\,\mathrm{d}s = \mathrm{E}\{Y^2(t)\}. \tag{4}$$

If $y(t)$ were a sample from an ergodic stationary process $Y(t)$, then the validity of (4)—the left member of which is actually a time average because of (3)—is guaranteed (with probability equal to unity) by the ergodic theorem.

To show that (3) produces an isomorphism, we must show that it preserves distances. We can denote the time-series obtained by taking the difference of two time-series $y_1(t)$ and $y_2(t)$ by $y(t)$ and, similarly, $Y(t) \triangleq Y_1(t) - Y_2(t)$. We need to show that

$$\mathrm{E}\{Y^2(t)\} = \langle y^2(t)\rangle. \tag{5}$$

It follows from (1), (3) and (4) that this is equivalent to showing that

$$\langle y^2(t+s)\rangle = \langle y^2(t)\rangle. \tag{6}$$

That is, we must show that the time-average is independent of time translation. This is verified under mild conditions on $y(t)$ in [30] (viz., that the autocorrelation $\langle y^2(t+\tau/2)y^2(t-\tau/2)\rangle$ exists and is continuous at $\tau=0$).

Examples of time-series $x(t)$ for which the preceding is valid for all measurement time-series $y(t)$ in the class considered include (with probability equal to unity) any of the sample paths of any ergodic stationary stochastic process. A class of examples that is specified independently of stochastic processes is presented in Section 5 (see also [23].)

To gain more insight into this isomorphism, we observe that by considering the particular time-series of measurements (again letting $M=2$ as an example)

$$z(t) \triangleq \begin{cases} 1, & x(t+t_1) < v \text{ and } x(t+t_2) < w, \\ 0, & \text{otherwise;} \end{cases} \tag{7}$$

in the metric space, we obtain a mean-square value $\langle z^2(t)\rangle$, which is equal to the mean value

$$\langle z(t)\rangle \triangleq \hat{F}_{x(t_1)x(t_2)}(v, w) \tag{8}$$

in this case, that has the interpretation of being the fraction-of-time that $x(t+t_1)<v$ and $x(t+t_2)<w$. That is, $\hat{F}_{x(t_1)x(t_2)}(v, w)$ is a valid joint cumulative probability distribution function [23]. Thus, the isomorphism reveals that we have a one-to-one correspondence between the probability distributions, such as $F_{X(t_1)X(t_2)}(v, w)$, and the fraction-of-time distributions, such as $\hat{F}_{x(t_1)x(t_2)}(v, w)$.

As an example, we consider the pulse-amplitude-modulated signal

$$x(t)=\sum_n a_n p(t-nT),$$

where $P(f)$, the Fourier transform of $p(t)$, is given by

$$P(f)=\begin{cases}[1+\cos(\pi f T)], & |f|\leqslant 1/T,\\ 0, & |f|>1/T.\end{cases}$$

A time-series representing this signal is shown in Fig. 1(a), and the measurement time-series $z(t)$ in (7) and its average value (8) are shown in Fig. 1(b) for $v=1/2$, $w=-1/2$, $t_1=0$ and $t_2=T/2$. This average value can be interpreted as one point on the surface above the v-w plane described by the function $\hat{F}_{x(t_1)x(t_2)}(v, w)$ for fixed t_1 and t_2.

The derivative of the cumulative joint probability distribution,

$$\frac{\partial^2}{\partial v \partial w}\hat{F}_{x(t_1)x(t_2)}(v, w)\triangleq \hat{f}_{x(t_1)x(t_2)}(v, w), \tag{9}$$

is a valid joint probability density function, Moreover, (7)–(9) yields the following characterization of time averages:

$$\langle g[x(t+t_1), x(t+t_2)]\rangle = \iint g(v, w)\hat{f}_{x(t_1)x(t_2)}(v, w)\,\mathrm{d}v\,\mathrm{d}w. \tag{10}$$

That is, the time-average of a measurement on a time-series can be interpreted as an expected value with respect to the fraction-of-time density. This

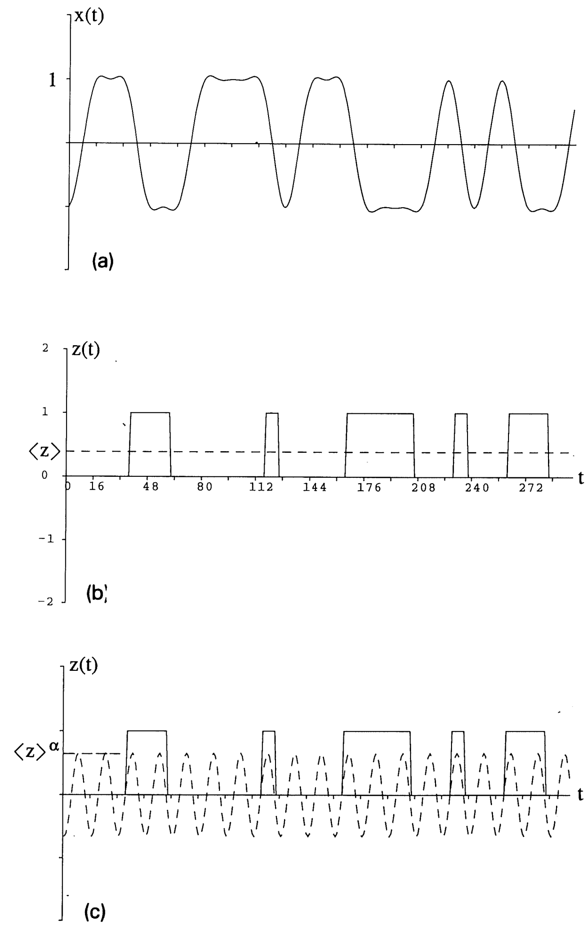

Fig. 1. Illustration of measurement time-series and its components: (a) pulse-amplitude-modulated signal $x(t)$ with period $T=16$; (b) measurement time-series $z(t)$ in (7), for $t_1=0$, $t_2=T/2$, $v=1/2$, $w=-1/2$, and its average value (constant component) $\langle z(t)\rangle=0.388$; (c) measurement time-series $z(t)$ in (7) and the real part of its sine-wave component $\langle z(t)\rangle^{\alpha}=0.0655$ with frequency $\alpha=1/T$ (two different scales are superimposed in this graph).

can be verified formally by substituting (7) into (8), the result of this into (9), and the result of this into (10); that is, by first reexpressing (7) as

$$z(t)=u[v-x(t+t_1)]u[w-x(t+t_2)], \tag{7'}$$

where $u(\;)$ is the unit step function

$$u(v)\triangleq\begin{cases}1, & v>0\\ 0, & v\leqslant 0;\end{cases}$$

we obtain from (7)-(9)

$$\iint g(v, w)\hat{f}_{x(t_1)x(t_2)}(v, w)\,\mathrm{d}v\,\mathrm{d}w$$
$$= \iint g(v, w)\frac{\partial^2}{\partial v \partial w}\langle u[v - x(t+t_1)] \times u[w - x(t+t_2)]\rangle\,\mathrm{d}v\,\mathrm{d}w$$
$$= \iint g(v, w)\langle \delta[v - x(t+t_1)] \times \delta[w - x(t+t_2)]\rangle\,\mathrm{d}v\,\mathrm{d}w$$
$$= \left\langle \iint g(v, w)\delta[v - x(t+t_1)] \times \delta[w - x(t+t_2)]\,\mathrm{d}v\,\mathrm{d}w \right\rangle$$
$$= \langle g[x(t+t_1), x(t+t_2)]\rangle, \tag{11}$$

where $\delta(v) = \mathrm{d}u(v)/\mathrm{d}v$ is the Dirac delta. This is, of course, only a formal manipulation since we have freely interchanged the order of execution of limits and integrals, not to mention our use of the Dirac delta.

The characterization (10) emphasizes the complete duality of concepts (and theory) that exists between time-averaged measurements on a single time-series and expected values of measurements on an ergodic stationary stochastic process. It enables us to conceive of a complete probabilistic model for a single time series and to apply probabilistic concepts and probability theory to single time-series. Moreover, the characterization (10) does not require the existence of the isomorphic stochastic process in (our explicit version of) Wold's isomorphism, since this characterization exists in its own right as long as the time averages it involves exist. Thus, it totally frees the mind from the abstract stochastic framework. This idea is fully developed in [23]. The objective in this paper is to generalize both Wold's isomorphism and the characterization (10) (for arbitrary finite dimension M) from stationary time-series to time-series that exhibit cyclostationarity with either a single period or multiple incommensurate periods.

3. Wold's isomorphism—generalized

If a persistent time-series of interest contains periodic structure with period T, we can obtain an isomorphism analogous to (our explicit version of) Wold's from the mapping

$$y(t, s) = y(t+s), \quad s = nT \text{ for } n = 0, \pm 1, \pm 2, \ldots. \tag{12}$$

In this case, (4) must be replaced with

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{s/T=-N}^{N} y^2(t, s) = \mathrm{E}\{Y^2(t)\}, \tag{13}$$

and (1) must be replaced with

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} y^2(t+nT) = \langle y^2(t)\rangle_T. \tag{14}$$

It is obvious that (12)-(14) satisfy the isomorphic property: $\mathrm{E}\{Y^2(t)\} = \langle y^2(t)\rangle_T$ provided only that $\langle y^2(t+s)\rangle_T = \langle y^2(t)\rangle_T$ for $s = nT$ analogous to (6). However, in this case, the isomorphic stochastic process is cycloergodic [4] (period-synchronized time-averages of random variables, such as (14), from the process converge with probability equal to unity) and cyclostationary with period T [8] (probability distributions are invariant to only translations that are integer multiples of the period T), rather than ergodic and stationary. For example, $\mathrm{E}\{Y^2(t+s)\} = \mathrm{E}\{Y^2(t)\}$ in general only for $s = nT$. Also, the fraction-of-time distributions are nonstationary in the same sense, that is, they are cyclostationary rather than stationary. For example,

$$\hat{F}_{x(t_1+nT)x(t_2+nT);T} \equiv \hat{F}_{x(t_1)x(t_2);T}, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{15}$$

where (using (7) as in (8))

$$\hat{F}_{x(t_1)x(t_2);T}(v, w) \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} z(t+nT). \tag{16}$$

If there is no underlying periodic structure with period T in $x(t)$, then the cyclostationarity of the model degenerates into stationarity:

$$\hat{F}_{x(t_1+t)x(t_2+t);T} \equiv \hat{F}_{x(t_1)x(t_2);T}, \quad -\infty < t < \infty. \tag{17}$$

But if there is indeed periodic structure present, then (17) is in general invalid except for $t = nT$ as in (15).

The cyclostationary fraction-of-time model (16) and the concept of cycloergodicity were in use explicitly in the field of communications at least as early as the mid 1950s (cf. [3]) and are currently being used explicitly in both analytical and experimental studies of the statistical behavior of communication systems (cf. [27]). However, the only attempt at a general theory of cycloergodicity, which accommodates multiple incommensurate periodicities, did not appear until recently [4].

Unfortunately, this straightforward method of generalizing Wold's isomorphism and fraction-of-time distributions does not work for time-series with multiple incommensurate periodicities. Only one periodicity can be reflected in the fraction-of-time model obtained in this way. Furthermore, this method does not apply to discrete time-series with only a single periodicity unless the period T is an integer multiple of the time increment T_0 between discrete-time samples in the series. In the next section, these limitations are circumvented by taking another approach. This alternative approach also circumvents the need for the existence of an isomorphic stochastic process.

4. Fraction-of-time probability—generalized

Let us now set aside the concept of a stochastic process, and pursue the dual concept of nonstochastic fraction-of-time probability. As briefly explained in Section 2 (and thoroughly explained and illustrated in [23]), we can in principle obtain a complete fraction-of-time probabilistic model for a persistent time-series in terms of the finite-order joint fraction-of-time distributions defined by

$$\hat{F}_{\boldsymbol{x}(t)}(\boldsymbol{v}) \triangleq \left\langle \prod_{m=1}^{M} u[v_m - x(t+s+t_m)] \right\rangle, \quad M = 1, 2, 3, \ldots, \tag{18}$$

where $\langle\,\rangle$ is the continuous average over the temporal phase parameter s and

$$\boldsymbol{x}(t) = \{x(t+t_1), x(t+t_2), \ldots, x(t+t_M)\}.$$

The corresponding finite-order joint fraction-of-time densities are given by

$$\hat{f}_{\boldsymbol{x}(t)}(\boldsymbol{v}) = \frac{\partial^M}{\partial v_1 \partial v_2 \cdots \partial v_M} \hat{F}_{\boldsymbol{x}(t)}(\boldsymbol{v}). \tag{19}$$

The average value of any (finite mean-square) finite dimensional measurement function

$$y(t) = g[\boldsymbol{x}(t)]$$

can be obtained using the following *fundamental theorem of averaging*:

$$\langle g[\boldsymbol{x}(t)]\rangle = \int g(\boldsymbol{v}) \hat{f}_{\boldsymbol{x}(t)}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v}. \tag{20}$$

Under mild conditions on $y(t)$ [30] (continuity of the autocorrelation of $y^2(t)$), all such averages are independent of time translation. That is, the fraction-of-time probabilistic model is stationary:

$$\hat{f}_{\boldsymbol{x}(t+s)} \equiv \hat{f}_{\boldsymbol{x}(t)}, \quad -\infty < s < \infty.$$

One way to generalize this idea from stationary models to models that exhibit cyclostationarity is to reinterpret the averaging operation as a constant-component-extraction operation, and then generalize this to sine-wave-component extraction. This is accomplished in the remainder of this section.

4.1. Constant-component extraction

The time-averaging operation

$$\langle y(t)\rangle = \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} y(t+s)\, \mathrm{d}s \tag{21}$$

extracts the constant component of its argument.

That is, the residual time-series

$$r(t) \triangleq y(t) - \langle y(t) \rangle$$

contains no constant component in the sense that it is orthogonal to all constant time-series,

$$\lim_{W\to\infty} \frac{1}{W} \int_{-W/2}^{W/2} r(t) c \, \mathrm{d}t = 0, \quad c = \text{constant}.$$

Furthermore, if we assume that the integral in (21) converges in temporal mean square,

$$\lim_{Z\to\infty} \lim_{W\to\infty} \frac{1}{W} \times \int_{-W/2}^{W/2} \left[\frac{1}{Z} \int_{-Z/2}^{Z/2} y(t+s) \, \mathrm{d}s - \langle y(t) \rangle \right]^2 \mathrm{d}t = 0,$$

to its pointwise limit $\langle y(t) \rangle$, then it can be shown [12, Chapter 5] that the spectral density of the time-averaged power, or time-average power spectral density, of the residual contains no spectral line (no Dirac delta) at zero frequency.

From this observation, we see that the theory of stationary fraction-of-time probability (or temporal probability) deals with nothing more than the constant components of measurements $y(t) = g[\boldsymbol{x}(t)]$ on a time-series $x(t)$. In fact, the time-average counterpart (20) of the fundamental theorem of expectation is really a *fundamental theorem of constant-component extraction.* Given knowledge of the Mth order fraction-of-time density function (or *temporal probability density function*) for a time-series $x(t)$, we can extract the constant component of any (finite mean-square) M-dimensional measurement function $g[\boldsymbol{x}(t)]$ by performing the integration (20).

As an example, the constant component of the measurement time-series (7) for the pulse-amplitude-modulated signal described in Section 2 is shown as the dashed line in Fig. 1(b).

4.2. Sine-wave-component extraction

For time-series with periodic structure, we are often interested not only in constant components of measurements, but also in sine-wave components. For example, in the study of modulated signals, the sine-wave components of the lag-product measurement

$$g[\boldsymbol{x}(t)] = x(t+\tau/2)x(t-\tau/2)$$

are quite important since they can be exploited for various signal processing tasks, such as timing synchronization [11], detection [16], time-difference estimation [20], source-direction estimation [29], and signal waveform extraction [1, 5] (cf. [12, Chapter 14]). Consequently, let us define the sine-wave extraction operation for frequency α as follows:

$$\langle y(t) \rangle^{\alpha} \triangleq \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} y(t+s) \times \exp(-\mathrm{i}2\pi\alpha s) \, \mathrm{d}s. \tag{22}$$

It can easily be shown (by substituting $r(t)$ into (22)) that the residual time-series

$$r(t) \triangleq y(t) - \langle y(t) \rangle^{\alpha}$$

contains no sine-wave component with frequency α in the sense that it is orthogonal to all sine waves of frequency α, regardless of their phase ϕ;

$$\lim_{W\to\infty} \frac{1}{W} \int_{-W/2}^{W/2} r(t) \exp(\mathrm{i}2\pi\alpha t + \mathrm{i}\phi) \, \mathrm{d}t = 0.$$

Furthermore, if we assume that the integral in (22) converges in temporal mean square to its pointwise limit, then it can be shown that the time-average power spectral density of the residual contains no spectral line at frequency α [12, Chapter 15].

By analogy with the definition (18) of the M-th order fraction-of-time distribution function, or *temporal distribution function,* for constant-component extraction, the Mth order temporal distribution function for sine-wave-component extraction is defined by

$$\hat{F}^{\alpha}_{\boldsymbol{x}(t)}(\boldsymbol{v}) \triangleq \left\langle \prod_{m=1}^{M} u[v_m - x(t+t_m)] \right\rangle^{\alpha}. \tag{23}$$

Although $\hat{F}^{\alpha}_{x(t)}(\ )$ is not a valid *probability* distribution function (except for $\alpha=0$, since $\langle\ \rangle^0 \equiv \langle\ \rangle$), it is a valid complex-valued distribution function. (It takes on values only in the unit disc in the complex plane and has a magnitude that is a nondecreasing function that ranges from zero to unity.) Moreover, it does yield a *fundamental theorem of sine-wave extraction* analogous to (20). That is, the sine-wave component with frequency α of any (finite mean-square) finite-dimensional function $g(\ )$ of $x(t)$ can be extracted by performing the integration

$$\langle g[x(t)]\rangle^{\alpha} = \int_{-\infty}^{\infty} g(v)\hat{f}^{\alpha}_{x(t)}(v)\,\mathrm{d}v, \tag{24}$$

where

$$\hat{f}^{\alpha}_{x(t)}(v) \triangleq \frac{\partial^M}{\partial v_1 \cdots \partial v_M}\hat{F}^{\alpha}_{x(t)}(v). \tag{25}$$

A formal verification of the validity of (24) follows by analogy with the formal verification (11) given in Section 2.

As an example, the real part of the sine-wave component, with frequency $\alpha = 1/T$, of the measurement time-series (7) for the pulse-amplitude-modulated signal described in Section 2 is shown in Fig. 1(c).

4.3. Periodic-component extraction

Although the sine-wave extraction operation does lead to a fundamental theorem of sine-wave extraction, it does not provide us with a temporal-probability model for the time-series. But we can obtain such a model by combining individual-sine-wave extraction operations into a composite-sine-wave extraction, or periodic-component extraction, operation as follows:

$$\langle y(t)\rangle_T \triangleq \sum_{\alpha} \langle y(t)\rangle^{\alpha}, \tag{26}$$

where the sum ranges over all integer multiples p of the fundamental frequency $1/T$ corresponding to the period T of interest ($\alpha = p/T$).

Another expression for (26) can be obtained from the following formal manipulation[2]:

$$\begin{aligned}\langle y(t)\rangle_T &\triangleq \sum_{p=-\infty}^{\infty} \lim_{Z\to\infty}\frac{1}{Z}\int_{-Z/2}^{Z/2} y(t+s) \\ &\qquad\times \exp(-\mathrm{i}2\pi ps/T)\,\mathrm{d}s \\ &= \lim_{Z\to\infty}\frac{1}{Z}\int_{-Z/2}^{Z/2} y(t+s) \\ &\qquad\times \sum_{p=-\infty}^{\infty}\exp(-\mathrm{i}2\pi ps/T)\,\mathrm{d}s \\ &= \lim_{z\to\infty}\frac{T}{Z}\int_{-Z/2}^{Z/2} y(t+s)\sum_{n=-\infty}^{\infty}\delta(s-nT)\,\mathrm{d}s \\ &= \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} y(t+nT).\end{aligned} \tag{27}$$

From this, we see that (26) is the Fourier-series expansion of (27).

Unfortunately, the periodic-component-extraction operation (27) cannot be applied to a discrete-time-series with sampling increment T_0 unless T and T_0 are commensurate. However, the alternative but equivalent operation (26) (with the integral in $\langle\ \rangle$ replaced by sum) can indeed be applied regardless of the values T_0 and T. The discrete-time counterpart of (26) is

$$\begin{aligned}\langle y_n\rangle_T &\triangleq \sum_{p\in[-T/2,T/2)} \lim_{K\to\infty}\frac{1}{2K+1} \\ &\qquad\times \sum_{k=-K}^{K} y_{n+k}\exp(-\mathrm{i}2\pi pk/T).\end{aligned} \tag{28}$$

This is pursued in Section 6.

The Mth order joint temporal-probability distribution for a time-series $x(t)$, based on the periodic-component-extraction operation (26)–(27), is defined by

$$\hat{F}_{x(t);T}(v) \triangleq \left\langle \prod_{m=1}^{M} u[v_m - x(t+t_m)]\right\rangle_T, \tag{29}$$

and it is cyclostationary (under mild conditions on the argument of $\langle\ \rangle_T$) in the sense that it varies

[2] A more rigorous derivation of this *synchronized averaging identity* (27) is given in [12, Chapter 10].

periodically in t with period T ((29) is identical to (16) for $M=2$). It yields the following *fundamental theorem of periodic-component extraction*:

$$\langle g[x(t)]\rangle_T = \int_{-\infty}^{\infty} g(\boldsymbol{v}) \hat{f}_{x(t);T}(\boldsymbol{v})\, d\boldsymbol{v}, \tag{30}$$

where

$$\hat{f}_{x(t);T}(\boldsymbol{v}) \triangleq \frac{\partial^M}{\partial v_1 \cdots \partial v_M} \hat{F}_{x(t);T}(\boldsymbol{v}). \tag{31}$$

A formal verification of the validity of (30) can be obtained by analogy with the formal verification (11) given in Section 2.

As an example, if $x(t)$ were a sample path from a cycloergodic cyclostationary stochastic process, then (29) would equal (with probability equal to unity) the stochastic-probability distribution for the process [4]. However, if $x(t)$ were a sample path from an ergodic stationary process, then (29) would yield the stationary distribution of that process, regardless of the value of T. Also, if $x(t)$ were a sample path from a cycloergodic cyclostationary process with period incommensurate with T, then (29) would yield the stationary component (average over the time-translation, or phase, parameter) of the cyclostationary stochastic-probability distribution (cf. [12, Chapter 15]). Analogous remarks apply to the class of Gaussian time-series that are defined independently of stochastic processes in Section 5.

We see that for a given time-series with periodic structure, we can in principle define either a stationary temporal-probability model based on the constant-component extraction operation $\langle\,\rangle$ or a cyclostationary temporal-probability model based on the periodic-component extraction operation $\langle\,\rangle_T$. Furthermore, we can obtain the former from the latter by time averaging since

$$\langle\langle y(t)\rangle_T\rangle = \langle y(t)\rangle. \tag{32}$$

The validity of (32) can be seen by substituting (22) into (26) and the result into the left member of (32), and then interchanging the order of the sum over $\{\alpha\}$ and the operation $\langle\,\rangle$.

As a result of (32), we obtain

$$\hat{F}_{x(t)}(\boldsymbol{v}) = \langle \hat{F}_{x(t);T}(\boldsymbol{v})\rangle = \lim_{Z\to\infty} \int_{-\infty}^{\infty} \hat{F}_{x(t+s);T}(\boldsymbol{v}) f(s; Z)\, ds, \tag{33}$$

where

$$f(s;Z) \triangleq \begin{cases} 1/Z, & |s|\leq Z/2, \\ 0, & |s|>Z/2. \end{cases}$$

We can interpret $\hat{F}_{x(t+s);T}(\boldsymbol{v})$ as a phase-conditional probability distribution and $f(s;Z)$ as the conditioning probability density. Then, from the definition of conditional probability, the integrand in (33) is a joint probability from which the marginal probability is obtained by integration over the phase parameter s. This technique of obtaining the stationary model from the cyclostationary model is called *phase randomization* and is useful when the periodic structure in the time-series is of no interest (cf. [7]).

4.4. Almost-periodic-component extraction

A function $a(t)$ that is the sum (or limit of partial sums) of periodic functions with incommensurate periods is said to be *almost periodic*[3] [6]. Such a function can be represented by a Fourier series

$$a(t) = \sum_{\alpha} a_\alpha \exp(i2\pi\alpha t) \tag{34}$$

(when this series converges), where α ranges over the integer multiples of each and every fundamental frequency $1/T$ corresponding to each and every fundamental period T. The Fourier coefficients in this series are given by

$$a_\alpha = \lim_{Z\to\infty} \frac{1}{Z} \int_{-Z/2}^{Z/2} a(s) \exp(-i2\pi\alpha s)\, ds. \tag{35}$$

If a time-series $x(t)$ exhibits periodic structure with multiple incommensurate periods (e.g., $x(t) = a(t)b(t)$, where $b(t)$ is a sample path from an ergodic stationary process), then we can define a distinct cyclostationary temporal-probability

[3] The generalized definition that includes discontinuous functions is needed here.

model for each period T, as well as a stationary temporal-probability model. Moreover, we can define a composite *almost cyclostationary model* that takes into account all periodic structure in the time-series. This is accomplished through use of the composite-sine-wave extraction, or almost-periodic-component extraction, operation defined by

$$\langle y(t)\rangle^{\{\alpha\}} \triangleq \sum_{\alpha} \langle y(t)\rangle^{\alpha}, \tag{36}$$

where in contrast to (26) the sum in (36) ranges over the integer multiples of each and every fundamental frequency $1/T$ associated with the sine-wave components of $y(t)$.

By decomposing the sum (36) into component sums, each of which ranges over only the integer multiples of one fundamental frequency, we obtain the identity

$$\langle y(t)\rangle^{\{\alpha\}} \equiv \langle y(t)\rangle_{\{T\}}, \tag{37}$$

where

$$\langle y(t)\rangle_{\{T\}} \triangleq \langle y(t)\rangle + \sum_{T} [\langle y(t)\rangle_T - \langle y(t)\rangle], \tag{38}$$

in which the sum ranges over all incommensurate periods $\{T\}$ associated with the periodic components of $y(t)$. The $\alpha = 0$ term $\langle y(t)\rangle \equiv \langle y(t)\rangle^0$ is subtracted from each periodic term since it would otherwise be included in the overall sum more than once.

Either (36) or (38) can be used to define an Mth order temporal-probability distribution through use of (23) and (29). Specifically, the definition

$$\hat{F}^{\{\alpha\}}_{x(t)}(\boldsymbol{v}) \triangleq \left\langle \prod_{m=1}^{M} u[v_m - x(t+t_m)] \right\rangle^{\{\alpha\}} \tag{39}$$

leads to

$$\hat{F}^{\{\alpha\}}_{x(t)}(\boldsymbol{v}) = \sum_{\alpha} \hat{F}^{\alpha}_{x(t)}(\boldsymbol{v}), \tag{40}$$

and the equivalent definition

$$\hat{F}_{x(t);\{T\}}(\boldsymbol{v}) \triangleq \left\langle \prod_{m=1}^{M} u[v_m - x(t+t_m)] \right\rangle_{\{T\}} \tag{41}$$

leads to

$$\hat{F}_{x(t);\{T\}}(\boldsymbol{v}) = \hat{F}_{x(t)}(\boldsymbol{v}) + \sum_{T} [\hat{F}_{x(t);T}(\boldsymbol{v}) - \hat{F}_{x(t)}(\boldsymbol{v})]. \tag{42}$$

Since it is by no means obvious that either (40) or (42) is indeed a valid probability distribution function, a proof of this proposition is given in Appendix A.

The almost cyclostationary temporal-probability model (40) or (42) for a time-series $x(t)$ characterizes all sine-wave components of all (finite mean-square) finite-dimensional functions $g(\,)$ of time translates $\{x(t+t_m)\}$. This follows from the *fundamental theorem of almost-periodic-component extraction*, which is embodied in the formula

$$\langle g[\boldsymbol{x}(t)]\rangle^{\{\alpha\}} = \int_{-\infty}^{\infty} g(\boldsymbol{v}) \hat{f}^{\{\alpha\}}_{x(t)}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v}. \tag{43}$$

As before, a formal verification of the validity of (43) can be obtained by analogy with the formal verification (11) given in Section 2.

As generalizations of the identity (32), we have the three identities

$$\langle\langle y(t)\rangle_{\{T\}}\rangle = \langle y(t)\rangle, \tag{44a}$$

$$\langle\langle y(t)\rangle_{\{T\}}\rangle_T = \langle y(t)\rangle_T, \tag{44b}$$

$$\langle\langle y(t)\rangle^{\{\alpha\}}\rangle^{\alpha} = \langle y(t)\rangle^{\alpha}. \tag{44c}$$

The identity (44b) reveals that we can obtain a cyclostationary model from the almost cyclostationary model by averaging:

$$\hat{F}_{x(t);T}(\boldsymbol{v}) = \langle \hat{F}_{x(t);\{T\}}(\boldsymbol{v})\rangle_T = \lim_{N\to\infty} \frac{1}{2N+1} \times \sum_{n=-N}^{N} \hat{F}_{x(t+nT);\{T\}}(\boldsymbol{v}). \tag{45}$$

By reexpressing (45) as

$$\hat{F}_{x(t);T}(\boldsymbol{v}) = \lim_{Z\to\infty} \int_{-\infty}^{\infty} \hat{F}_{x(t+s);\{T\}}(\boldsymbol{v}) f_T(s;Z)\, \mathrm{d}s, \tag{46}$$

where $f_T(s; Z)$ is defined by

$$f_T(s; Z) \triangleq \begin{cases} \dfrac{1}{1+[Z/T]} \sum_n \delta(s-nT), & |s| \leq Z/2, \\ 0, & \text{otherwise,} \end{cases} \tag{47}$$

in which $[Z/T]$ is the integer part of Z/T, we can interpret $\hat{F}_{x(t+s);\{T\}}(v)$ as a phase-conditional probability distribution and $f_T(s; Z)$ as the conditioning probability density, Then, from the definition of conditional probability, the integrand in (46) is a joint probability from which the marginal is obtained by integration over the phase parameter s. This is a phase randomization procedure. Similarly, identity (44a) reveals that we can obtain the stationary model from the almost cyclostationary model by phase randomization, as in (33):

$$\hat{F}_{x(t)}(v) = \langle \hat{F}_{x(t);\{T\}}(v) \rangle = \lim_{Z\to\infty} \int_{-\infty}^{\infty} \hat{F}_{x(t+s);\{T\}}(v) f(s, Z)\, \mathrm{d}s, \tag{48}$$

where

$$f(s; Z) \triangleq \begin{cases} 1/Z, & |s| \leq Z/2, \\ 0, & |s| > Z/2. \end{cases}$$

This subsection is concluded with a few words about terminology. If $\hat{F}_{x(t);T} \equiv \hat{F}_{x(t)} \not\equiv 0$ for all periods T, then $x(t)$ is said to be a *purely stationary* time-series (of order M), and if $\hat{F}_{x(t);T} \not\equiv \hat{F}_{x(t)}$ for one and only one fundamental period T, then $x(t)$ is said to be *purely cyclostationary* (of order M). Otherwise, $x(t)$ is said to be *almost cyclostationary* (of order M), assuming $\hat{F}_{x(t);\{T\}} \not\equiv 0$.

4.5. Temporal independence

By analogy with the definition of statistical independence for stochastic processes, two time-series $x_1(t)$ and $x_2(t)$ are said to be *temporally independent* if and only if all their joint temporal-probability distributions factor according to

$$\hat{F}_{x_1(t),x_2(t);\{T\}}(v, w) = \hat{F}_{x_1(t);\{T\}}(v)\hat{F}_{x_2(t);\{T\}}(w) \tag{49}$$

for all orders M and N and all sets of time points $\{t_1, t_2, \ldots, t_M\}$ and $\{s_1, s_2, \ldots, s_N\}$ for $x_1(t)$ and $x_2(t)$, respectively. It follows from the almost-periodic-component-extraction definition (41) and the fundamental theorem of almost-periodic-component extraction (43) that (49) implies that

$$\langle g_1[x_1(t)]g_2[x_2(t)]\rangle_{\{T\}} = \langle g_1[x_1(t)]\rangle_{\{T\}}\langle g_2[x_2(t)]\rangle_{\{T\}}, \tag{50}$$

which simply means that the almost periodic component of the product of any (finite mean-square) finite-dimensional functions $g_1(\,)$ and $g_2(\,)$, of the time-series $x_1(t)$ and $x_2(t)$, is equal to the product of corresponding almost periodic components. Since the almost periodic component of the product of an almost periodic function and any other function is the product of the almost periodic function and the almost periodic component of the other function, it follows from (50) that every almost periodic (not just almost cyclostationary) time-series is temporally independent of every time-series, including itself. The stochastic-probability counterpart of this is that every stochastic process that equals any deterministic function with probability equal to one is statistically independent of every stochastic process, including itself.

It must be stressed that temporal independence relative to a temporal-probability model for a given time-series that incorporates a certain set of sine-wave frequencies $S_1 = \{\alpha\}$ does not imply temporal independence relative to a temporal-probability model for the same time-series that incorporates a proper subset $S_2 \subset S_1$ of sine-wave frequencies. This follows from the fact that conditional independence does not imply unconditional independence. Thus, for example, two time-series can be temporally independent relative to their cyclostationary temporal-probability model while being temporally dependent relative to their stationary temporal-probability model.

4.6. Examples

Product modulation

Let us consider the product time-series

$$x(t) = a(t)b(t), \tag{51}$$

where $a(t)$ is a positive-valued almost periodic function. Then $a(t)$ is temporally independent of $b(t)$ and can, therefore, be treated as a time-variant scale factor in which case we have[4]

$$\hat{f}_{x(t);T}(\boldsymbol{v}) = \frac{1}{a(t+t_1)\cdots a(t+t_M)} \times \hat{f}_{b(t)}\left(\frac{v_1}{a(t+t_1)}, \ldots, \frac{v_M}{a(t+t_M)}\right). \quad (52)$$

If we desire the stationary temporal-probability model, we can simply average (52) as in (33), or if we desire a cyclostationary model corresponding to a specific period T, we can simply synchronously average (52) as in (45). The two time-series $a(t)$ and $b(t)$ are not temporally independent with respect to the stationary model or the cyclostationary models unless $a(t)$ is constant or periodic (period $= T$), respectively.

Almost periodic time-series

Let us consider the almost periodic time-series

$$x(t) = a(t) = \sum_\alpha a_\alpha \exp(i2\pi\alpha t). \quad (53)$$

The event indicator function

$$z(t) \triangleq \prod_{m=1}^{M} u[v_m - x(t+t_m)]$$

is almost periodic in this case. Thus, it equals its own almost periodic component,

$$\hat{F}_{x(t);\{T\}}(\boldsymbol{v}) = z(t).$$

Formal differentiation yields

$$\hat{f}_{x(t);\{T\}}(\boldsymbol{v}) = \prod_{m=1}^{M} \delta[v_m - x(t+t_m)]. \quad (54)$$

Using the scaling property of the Dirac delta, $\delta(v-a) = (1/|a|)\delta(v/a - 1)$, shows that this result agrees with the result of the previous example for the special case $b(t) \equiv 1$, in which case

$$f_{b(t)}(\boldsymbol{w}) = \prod_{m=1}^{M} \delta(w_m - 1).$$

If we desire a stationary model for $x(t)$, then we obtain a nondegenerate temporal-probability density by substituting (54) into (33). For instance, if $x(t) = a_0 + a_1 \sin \omega_1 t$, then it can be shown that

$$\hat{f}_{x(t)}(v) = \begin{cases} \pi[a_1^2 - (v-a_0)^2]^{-1/2}, & |v-a_0| < |a_1|, \\ 0, & \text{otherwise}. \end{cases} \quad (55)$$

It is clear from this simple example that very different temporal-probability models can be obtained from the same time-series, depending on how the periodic structure is dealt with. To illustrate this, the first-order cyclostationary model (54) (with $M = 1$) for this sine-wave-plus-a-constant time-series is graphed in Fig. 2(a), and the corre-

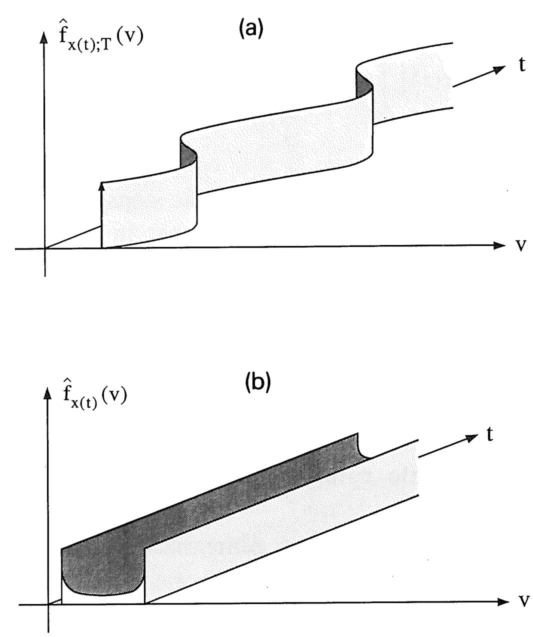

Fig. 2. Graphs of first order probability density functions: (a) for the cyclostationary model (54); (b) for the stationary model (55).

[4] This result is derived more methodically in [12, Chapter 15].

sponding stationary model (55) is graphed in Fig. 2(b).

4.7. Generalized fundamental theorem of almost-periodic-component extraction

For some applications, measurement functions change with time; that is, $g[x(t)]$ becomes $g[t, x(t)]$. Fortunately, the fundamental theorem of almost-periodic-component extraction, (43), can be generalized to accommodate almost periodic time-variation of $g(t,\ )$ with t. To demonstrate this, we proceed as follows with the right member of (43), generalized by replacement of $g(\ )$ with $g(t,\ )$, to obtain

$$\int_{-\infty}^{\infty} g(t, \boldsymbol{v}) f_{\boldsymbol{x}(t)}^{\{\alpha\}}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{v}$$

$$= \int_{-\infty}^{\infty} g(t, \boldsymbol{v}) \left\langle \prod_{m=1}^{M} \delta[v_m - x(t+t_m)] \right\rangle^{\{\alpha\}} \mathrm{d}\boldsymbol{v}$$

$$= \int_{-\infty}^{\infty} \left\langle g(t, \boldsymbol{v}) \prod_{m=1}^{M} \delta[v_m - x(t+t_m)] \right\rangle^{\{\alpha\}} \mathrm{d}\boldsymbol{v}$$

$$= \int_{-\infty}^{\infty} \left\langle g[t, \boldsymbol{x}(t)] \prod_{m=1}^{M} \delta[v_m - x(t+t_m)] \right\rangle^{\{\alpha\}} \mathrm{d}\boldsymbol{v}$$

$$= \left\langle g[t, \boldsymbol{x}(t)] \int_{-\infty}^{\infty} \delta[v_m - x(t+t_m)]\, \mathrm{d}\boldsymbol{v} \right\rangle^{\{\alpha\}}$$

$$= \langle g[t, \boldsymbol{x}(t)] \rangle^{\{\alpha\}}. \tag{56}$$

In (56), we have followed the same steps as in the formal verification (11), except that in order to obtain the second equality in (56), we invoked the result (cf. Section 4.5) that the almost periodic component of the product of any time-series ($\sum_{m=1}^{M} \delta[v_m - x(t+t_m)]$ in this case) and an almost periodic time-series ($g(t, \boldsymbol{v})$ in this case) is the product of the almost periodic time-series and the almost periodic component of the other time-series.

It is emphasized that generalizations of the fundamental theorems of component extraction (20), (24) and (30) that are analogous to (56) are, in general, not valid. That is, $g(\ )$ must, in general, be time-invariant in (20) and (24), and it must be periodic with period T in (30).

5. Gaussian almost cyclostationary temporal-probability models

In this section, we demonstrate that fraction-of-time probability models exist, independently of stochastic processes, by specifying a particular class of models without any reference to stochastic processes.

5.1. Real-valued time-series

An almost cyclostationary time-series $x(t)$ is defined to be *Gaussian* if and only if for every positive integer M and every M time points $\{t_1, t_2, \ldots, t_M\}$, every linear combination of the M time translates $\{x(t+t_1), x(t+t_2), \ldots, x(t+t_M)\}$, say

$$y(t) = \sum_{m=1}^{M} \omega_m x(t+t_m) = \boldsymbol{\omega}'\boldsymbol{x}(t), \tag{57}$$

has a first-order Gaussian temporal-probability density defined in terms of $\langle\ \rangle_{\{T\}}$:

$$\hat{f}_{y(t);\{T\}}(v) = \frac{1}{\hat{\sigma}(t)\sqrt{2\pi}} \times \exp\left\{\frac{-[v-\hat{\mu}(t)]^2}{2\hat{\sigma}^2(t)}\right\}. \tag{58}$$

It can be shown (cf. [8, Chapter 2]) using (58) that

$$\hat{\mu}(t) = \langle y(t)\rangle_{\{T\}}, \tag{59a}$$

$$\hat{\sigma}(t) = [\langle [y(t) - \hat{\mu}(t)]^2\rangle_{\{T\}}]^{1/2}, \tag{59b}$$

which are the almost periodically time-variant temporal mean and temporal standard deviation of the time-series $y(t)$. From this definition, it is easily shown [12, Chapter 15] that the joint temporal characteristic function for $x(t)$, defined by

$$\hat{\Psi}_{\boldsymbol{x}(t)}(\boldsymbol{\omega}) \triangleq \langle \exp[\mathrm{i}\, \boldsymbol{\omega}'\boldsymbol{x}(t)]\rangle_{\{T\}}, \tag{60}$$

is given by

$$\hat{\Psi}_{\boldsymbol{x}(t)}(\boldsymbol{\omega}) = \exp\{\mathrm{i}\boldsymbol{\omega}'\hat{\boldsymbol{M}}_{\boldsymbol{x}(t)} - \boldsymbol{\omega}'\hat{\boldsymbol{K}}_{\boldsymbol{x}(t)}\boldsymbol{\omega}\}, \tag{61}$$

where the M-vector $\hat{\boldsymbol{M}}_{\boldsymbol{x}(t)}$ has pth element

$$\hat{\boldsymbol{M}}_{\boldsymbol{x}(t)}(p) \triangleq \langle x(t+t_p)\rangle_{\{T\}}, \tag{62}$$

and the $M \times M$ matrix $\hat{\boldsymbol{K}}_{x(t)}$ has pq-th element

$$\hat{\boldsymbol{K}}_{x(t)}(p, q) \triangleq \langle [x(t+t_p) - \hat{\boldsymbol{M}}_{x(t)}(p)] \times [x(t+t_q) - \hat{\boldsymbol{M}}_{x(t)}(q)] \rangle_{\{T\}}, \quad (63)$$

which are the almost periodically time-variant temporal means and temporal covariances of $x(t)$, respectively. From the joint temporal characteristic function, we can obtain the joint temporal-probability density by M-dimensional Fourier transformation (cf. [8, Chapter 2]). Also, we can obtain all the temporal moments for $x(t)$ by differentiation of $\hat{\Psi}_{x(t)}(\boldsymbol{\omega})$ (as is well-known in probability theory, cf. [8, Chapter 2]), and these moments are given explicitly in terms of (62) and (63) by Isserlis' formula [8, Chapter 5]. Thus, the temporal-probability model for a Gaussian time-series is fully specified by specifying only the temporal mean (62) and temporal covariance (63).

This particular definition of a Gaussian time-series makes it obvious that the Gaussian property is preserved under linear transformation such as (57). That is, given that $x(t)$ is a Gaussian time-series, so is $y(t)$ (not just to first order as in (58)). By taking the limit as $M \to \infty$, this property can be used to show that continuous-time time-variant linear transformations such as

$$y(t) = \int_{-\infty}^{\infty} h(t, u) x(u) \, \mathrm{d}u, \quad (64)$$

also preserve the Gaussian property of an almost cyclostationary time-series, provided that $h(t, t-\tau)$ is almost periodic in t for each τ.

From the preceding, we see that the theory of Gaussian almost cyclostationary time-series is completely dual to the theory of Gaussian stochastic processes that are almost cyclostationary or, as special cases, cyclostationary or stationary (cf. [8]). However, it is emphasized that these three types of temporal-probability models are mutually exclusive. That is, if the almost cyclostationary model for $x(t)$ is Gaussian (and is not degenerate in the sense that it is not purely cyclostationary or purely stationary), then neither the cyclostationary models nor the stationary model for $x(t)$ can be Gaussian (because uniform mixtures of nonidentical Gaussian variates are non-Gaussian—cf. (33) and (46)). This fact illustrates that the ubiquitous assumption that time-series can be modeled as Gaussian can be contradictory especially when phase randomization techniques (such as (33) and (46)) are used as is often done with stochastic process models for modulated signals [7].

As an example of one of the pitfalls of inappropriate Gaussian modelling, it is shown in [5; 12, Chapter 15] that the measurement time-series at the output of a running (sliding-window) spectrum analyzer, with a Gaussian almost cyclostationary time-series that does not exhibit spectral lines at the input, exhibits spectral lines that cannot be predicted using the stationary Gaussian model. Furthermore, it is shown that the variability of the measurement time-series can be substantially different from that predicted with the stationary Gaussian model [5; 10; 12, Chapter 15]. Other pitfalls associated with phase-randomization techniques are described in [13].

Example: modulated Gaussian time-series

It follows from the preceding discussion that if the time-series factor $b(t)$ in the product modulation example in Section 4 is Gaussian, then so is the modulated time-series $x(t)$ in (51). The temporal-probability density for $b(t)$ can be obtained from (61)–(63) (with $x(t)$ there replaced by $b(t)$) and the temporal-probability density for $x(t)$ can be obtained from (52).

5.2. Complex-valued time-series

Analysis in the field of signal processing is often simplified considerably through the use of complex envelope and analytic signal representations. Consequently, the models described up to this point need to be generalized from real-valued to complex-valued time-series. In principle, this requires nothing more than reinterpreting complex-valued time-series as ordered pairs of real time-series, in which the elements of the pair are the real and imaginary parts. Nevertheless, it is shown in [5] that this process of reinterpretation can be handled in a particularly convenient way through the use of the unitary transformation

$$J \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -\mathrm{i}I & \mathrm{i}I \end{bmatrix} = J^{-\mathrm{H}}, \tag{65}$$

where I is the $M \times M$ identity matrix and $J^{-\mathrm{H}}$ is the Hermitian transpose of the inverse of J. The transformation, when applied to the complex extended vector

$$\vec{x}(t) \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} x(t) \\ x^*(t) \end{bmatrix} \tag{66}$$

consisting of $x(t)$ and its conjugate $x^*(t)$, produces the real extended vector

$$\tilde{x}(t) \triangleq \begin{bmatrix} x_{\mathrm{real}}(t) \\ x_{\mathrm{imag}}(t) \end{bmatrix}, \tag{67}$$

that is,

$$\tilde{x}(t) = J\vec{x}(t) \tag{68}$$

and

$$\vec{x}(t) = J^{\mathrm{H}}\tilde{x}(t). \tag{69}$$

The required generalization of the temporal-probability models for real-valued time-series described in the previous sections to the corresponding models for complex-valued time-series is carried out in [5]. Only the special case of Gaussian complex-valued time-series is briefly discussed here. It can be shown that the characteristic functions for the extended vectors are related by

$$\hat{\Psi}_{\vec{x}(t)}(\omega) = \hat{\Psi}_{\tilde{x}(t)}(J^* \omega), \tag{70}$$

and for a Gaussian time-series (real and imaginary parts jointy Gaussian) this yields

$$\hat{\Psi}_{\vec{x}(t)}(\omega) = \exp\{-\tfrac{1}{2}\omega' \hat{K}_{\vec{x}(t)\vec{x}^*(t)}\omega + \mathrm{i}\omega' \hat{M}_{\vec{x}(t)}\}, \tag{71}$$

where

$$\hat{K}_{\vec{x}(t)\vec{x}^*(t)} \triangleq \frac{1}{2} \begin{bmatrix} \hat{K}_{x(t)x^*(t)} & \hat{K}_{x(t)} \\ \hat{K}^*_{x(t)} & \hat{K}^*_{x(t)x^*(t)} \end{bmatrix} \tag{72}$$

and

$$\hat{M}_{\vec{x}(t)} \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{M}_{x(t)} \\ \hat{M}^*_{x(t)} \end{bmatrix}. \tag{73}$$

In (72), $\hat{K}_{x(t)}$ and $\hat{K}_{x(t)x^*(t)}$ are defined by

$$\hat{K}_{x(t)} \triangleq \langle x(t)x^{\mathrm{H}}(t)\rangle_{\{T\}} - \hat{M}_{x(t)}\hat{M}^{\mathrm{H}}_{x(t)}, \tag{74}$$

$$\hat{K}_{x(t)x^*(t)} \triangleq \langle x(t)x'(t)\rangle_{\{T\}} - \hat{M}_{x(t)}\hat{M}'_{x(t)}. \tag{75}$$

In the Gaussian stochastic-probability models typically found in the literature (e.g., [24–26, 28] it is assumed that the cross-covariance (75) is zero,

$$\hat{K}_{x(t)x^*(t)} \equiv 0, \tag{76}$$

in which case the model is completely specified by only $\hat{K}_{x(t)}$ and $\hat{M}_{x(t)}$. However, it is shown in [5] that assumption (76) is invalid for various Gaussian almost cyclostationary time-series. This has an important impact on the study of almost cyclostationary Gaussian time-series. For example, the cross-covariance (75) plays an important role in the power spectral density formulas for Rice's representation (cf. (33)–(35) in [14]). Also, it plays an important role in the theory of optimum linear filtering, as described in [5].

6. Discrete-time temporal probability

The discrete-time counterparts to the temporal probability models defined in Section 4 are based on the following four definitions of component-extraction operations:

Constant component:

$$\langle y_n \rangle \triangleq \lim_{K\to\infty} \frac{1}{2K+1} \sum_{k=-K}^{K} y_{n+k}, \tag{77}$$

by analogy with (21).

Sine-wave component:

$$\langle y_n \rangle^{\alpha} \triangleq \lim_{K\to\infty} \frac{1}{2K+1} \sum_{k=-K}^{K} y_{n+k} \times \exp(-\mathrm{i}2\pi\alpha k), \tag{78}$$

by analogy with (22).

Periodic component:

$$\langle y_n \rangle_T \triangleq \sum_{p\in[-T/2,T/2)} \langle y_n \rangle^{p/T}, \tag{79}$$

by analogy with (26).

Almost periodic component:

$$\langle y_n\rangle^{\{\alpha\}} \triangleq \sum_{\alpha\in[-1/2,1/2)} \langle y_n\rangle^{\alpha}$$
$$\equiv \langle y_n\rangle + \sum_T [\langle y_n\rangle_T - \langle y_n\rangle]$$
$$\triangleq \langle y_n\rangle_{\{T\}}, \tag{80}$$

by analogy with (36)-(38).

Specifically, the stationary model is defined by

$$\hat{F}_{x_n}(\boldsymbol{v}) \triangleq \left\langle \prod_{m=1}^{M} u[v_m - x_{n+k+n_m}] \right\rangle, \tag{81}$$

where $\langle\,\rangle$ is the discrete-time average over the temporal phase parameter k, by analogy with (18); the distribution for sine-wave component extraction is defined by

$$\hat{F}^{\alpha}_{x_n}(\boldsymbol{v}) \triangleq \left\langle \prod_{m=1}^{M} u[v_m - x_{n+k+n_m}] \right\rangle^{\alpha}, \tag{82}$$

by analogy with (23); the cyclostationary model is defined by

$$\hat{F}_{x_n;T}(\boldsymbol{v}) \triangleq \left\langle \prod_{m=1}^{M} u[v_m - x_{n+k+n_m}] \right\rangle_T, \tag{83}$$

by analogy with (29); and the almost cyclostationary model is defined by

$$\hat{F}^{\{\alpha\}}_{x_n}(\boldsymbol{v}) \triangleq \left\langle \prod_{m=1}^{M} u[v_m - x_{n+k+n_m}] \right\rangle^{\{\alpha\}},$$
$$\equiv \left\langle \prod_{m=1}^{M} u[v_m - x_{n+k+n_m}] \right\rangle_{\{T\}}$$
$$\triangleq \hat{F}_{x_n;\{T\}}(\boldsymbol{v}) \tag{84}$$

by analogy with (39) and (41).

It follows by analogy that the discrete-time counterparts to the fundamental theorems of component extraction, (20), (24), (30), (43) and (56) are valid.

When a discrete time-series y_n is obtained by time sampling a continuous time-series $y(t)$,

$$y_n = y(t)|_{t=n}, \tag{85}$$

we can relate the discrete-time component extraction operations (77)-(80) to their continuous-time counterparts. Specifically, we can show that

$$\langle y_n\rangle^{\alpha} = \left[\sum_{p=-\infty}^{\infty} \langle y(t)\rangle^{\alpha+p}\right]_{t=n}, \tag{86}$$

which reflects the well-known aliasing phenomenon. It follows that

$$\langle y_n\rangle^{\alpha} = \langle y(t)\rangle^{\alpha}|_{t=n} \tag{87a}$$

is valid only if $y(t)$ is bandlimited to less than half the sampling rate of unity, so that

$$\langle y(t)\rangle^{\alpha} \equiv 0 \quad \text{for } \alpha \geqslant 1/2,$$

or if

$$\langle y(t)\rangle^{\alpha+p} \equiv 0 \quad \text{for all } p \neq 0 \tag{87b}$$

happens to hold for some particular α. However, even if a time-series $x(t)$ is bandlimited, some measurement time-series $y(t)$, such as

$$y(t) = \prod_{m=1}^{M} u[v_m - x(t+t_m)],$$

will be nonbandlimited (because of the discontinuous behaviour of $u(\,)$ in this instance). Thus, in general, (87a) does not hold for all α.

It follows from (86) with $\alpha = 0$ that

$$\langle y_n\rangle = \left[\sum_{p=-\infty}^{\infty} \langle y(t)\rangle^{p}\right]_{t=n}. \tag{88}$$

Thus, again

$$\langle y_n\rangle = \langle y(t)\rangle|_{t=n} \tag{89a}$$

is valid only if

$$\langle y(t)\rangle^{p} \equiv 0 \quad \text{for } p \neq 0. \tag{89b}$$

It follows from the definition (79) and the property (86) that

$$\langle y_n\rangle_T = \left[\sum_{p\in[-T/2,T/2)} \sum_{q=-\infty}^{\infty} \langle y(t)\rangle^{p/T+q}\right]_{t=n}. \tag{90}$$

Thus,

$$\langle y_n\rangle_T = \langle y(t)\rangle_T|_{t=n} \tag{91a}$$

is valid only if either $y(t)$ is bandlimited to less than half the sampling rate of unity, or if

$$T = \text{integer}. \tag{91b}$$

In spite of the general invalidity of (87a), (89a) and (91a), we do have the general identity

$$\langle y_n \rangle^{\{\alpha\}} = \langle y(t) \rangle^{\{\alpha\}}|_{t=n}. \tag{92}$$

It follows from (92) that the discrete-time almost cyclostationary temporal probability model can always be obtained from the continuous-time model simply by time-sampling

$$\hat{F}_{x_n}^{\{\alpha\}}(\boldsymbol{v}) \equiv \hat{F}_{x(t)}^{\{\alpha\}}(\boldsymbol{v})|_{t=n}. \tag{93}$$

Unfortunately, the same simple relation does not hold in general for the cyclostationary and stationary models because of the general invalidity of (89a) and (91a).

7. Conclusion

The appropriateness and utility of probabilistic models that exhibit cyclostationarity for applications involving modulated signals, which arise in communications, radar and telemetry systems, is becoming widely accepted. However, the abstract concept of a stochastic process is not always desirable or appropriate. In this paper, it is shown how in principle to obtain nonstochastic temporal-probability models for single time-series that exhibit cyclostationarity with either a single period or multiple incommensurate periods. This renders probabilistic concepts and the theory and method of probability applicable to single time-series without the need for introducing the conceptual artifice of an ensemble of random samples and an associated stochastic process defined on an abstract probability space. The utility of temporal-probability models that exhibit cyclostationarity is amply demonstrated in [12, Part II] where a comprehensive theory of spectral correlation measurements on single time-series is developed. Spectral correlation is a characteristic property of second-order cyclostationarity. Numerous applications of spectral correlation to timing synchronization, detection, parameter estimation (including time-difference and source-direction) and waveform extraction are described in [12, Chapter 14] (see also [1; 5; 8, Chapter 12; 11; 16; 20; 29]), and explicit formulas for almost periodically time-variant autocorrelation functions and their corresponding spectral correlation density functions are derived for many types of modulated signals in [15; 22; 12, Chapter 12]. Tutorial treatments of spectral correlation theory and its applications are given in [9, 19], which can be viewed as the wide-sense theoretical counterpart of the strict-sense theory of nonstochastic modelling of time-series that exhibit cyclostationarity, which is presented in this paper. The utility of temporal-probability models that exhibit higher-than-second-order cyclostationarity is demonstrated in [2], where they are used to develop new methods for identification of nonlinear systems, and in [17, 21], where the theory of cyclic cumulants and cyclic polyspectra is introduced and applied to the problem of characterizing nonlinear spectral-line regenerators.

Appendix A

In order to prove that the almost cyclostationary temporal probability model (40) is indeed a valid M-th order probability distribution function, we must prove that

(1) it is nonnegative:

$$F_{x(t)}^{\{\alpha\}}(\boldsymbol{v}) \geq 0 \quad \text{for all } \boldsymbol{v};$$

(2) it is nonincreasing:

$$F_{x(t)}^{\{\alpha\}}(\boldsymbol{v}) \geq F_{x(t)}^{\{\alpha\}}(\boldsymbol{u}) \quad \text{if } \boldsymbol{v} \geq \boldsymbol{u}$$

(i.e., if $v_m \geq u_m$ for $m = 1, 2, 3, \ldots, M$); and

(3) it is consistent in the sense that the probability distribution function for any subset of the variables $\{x(t+t_m) \,|\, m = 1, 2, 3, \ldots, M\}$ can be obtained by letting $v_m \to \infty$ in $F_{x(t)}^{\{\alpha\}}(\boldsymbol{v})$ for all m not contained in the index set for this subset.

PROOF

(1) $F^{\{\alpha\}}_{x(t)}(\mathbf{v})$ is the almost periodic (a.p.) component, call it $p(t)$, of the zero-one indicator function

$$z(t)=\prod_{m=1}^{M} u[v_m - x(t+t_m)],$$

that is,

$$p(t)=\sum_{\alpha} \langle z(t)\rangle^{\alpha}.$$

We want to show that $p(t)\equiv F^{\{\alpha\}}_{x(t)}(\mathbf{v})$ is nonnegative. Let $r(t)\triangleq z(t)-p(t)$ be the residual, which contains no a.p. component. Then we have

$$\begin{aligned}0\leqslant z(t)&=p(t)+r(t)\\ &=[p(t)+q(t)]-q(t)+r(t),\end{aligned} \tag{A.1}$$

where $q(t)$ is the magnitude of the negative part of $p(t)$:

$$q(t)=g[p(t)]\triangleq\begin{cases}0, & p(t)\geqslant 0\\ -p(t), & p(t)<0.\end{cases}$$

We let $y(t)$ be the zero-one indicator function

$$y(t)=u[q(t)]\geqslant 0,$$

where

$$u(v)\triangleq\begin{cases}1, & v>0\\ 0, & v\leqslant 0.\end{cases}$$

Then since $p(t)+q(t)=0$ when $y(t)\neq 0$, and $q(t)y(t)=q(t)$, the inequality

$$0\leqslant y(t)[p(t)+q(t)]-y(t)q(t)+y(t)r(t)$$

obtained from (A.1) by multiplying by $y(t)$ reduces to

$$0\leqslant -q(t)+y(t)r(t),$$

which is equivalent to

$$y(t)r(t)\geqslant q(t)\geqslant 0. \tag{A.2}$$

Since $q(t)=g[p(t)]$ is a memoryless function of the a.p. function $p(t)$, then $q(t)$ also is an a.p., function and, since it is nonnegative, its average value must be strictly positive unless it is identically zero, $q(t)\equiv 0$. It follows from (A.2) that

$$\langle y(t)r(t)\rangle>0 \tag{A.3}$$

if $q(t)\not\equiv 0$.

But, since $y(t)$ is a memoryless function $u[\ ]$ of an a.p. function $q(t)$, then it is also a.p. (in the general sense that allows for step discontinuities). Consequently, (A.3) reveals that $r(t)$ contains an a.p. component if $q(t)\not\equiv 0$.

But since $r(t)$ by definition contains no a.p. component, then by contradiction we must have $q(t)\equiv 0$. Hence $p(t)=F^{\{\alpha\}}_{x(t)}(\mathbf{v})$ is nonnegative.

(2) We want to show that $F^{\{\alpha\}}_{x(t)}(\mathbf{v})=p(t)$ is a nondecreasing function of $\mathbf{v}$. When the argument $\mathbf{v}=\mathbf{v}_1$ is increased to $\mathbf{v}_2$, the corresponding indicator function $z(t)\triangleq z_2(t)$ will be non-zero over a superset of the set of values of t for which $z(t)\triangleq z_1(t)$ was non-zero before $\mathbf{v}$ was increased. Therefore

$$z_2(t)=z_1(t)+z_{12}(t),$$

where $z_{12}(t)$ is a zero-one function with support that is disjoint from the support of $z_1(t)$. The a.p. component of $z_2(t)$ can therefore be expressed as

$$F^{\{\alpha\}}_{x(t)}(\mathbf{v}_2)=p_2(t)=p_1(t)+p_{12}(t),$$

where

$$F^{\{\alpha\}}_{x(t)}(\mathbf{v}_1)=p_1(t)$$

and where $p_{12}(t)\geqslant 0$ as proved in (1) above. Thus

$$F^{\{\alpha\}}_{x(t)}(\mathbf{v}_2)\geqslant F^{\{\alpha\}}_{x(t)}(\mathbf{v}_1).$$

Hence $F^{\{\alpha\}}_{x(t)}(\ )$ is nondecreasing.

(3) We want to show that when $v_m\to\infty$ for m in some subset S_0 of the set $S=\{1,2,3,\ldots,M\}$, then $F^{\{\alpha\}}_{x(t)}(\mathbf{v})$ reduces to the joint probability distribution for v_m for m in the complement $\bar{S}_0$ of S_0 relative to S. Since $u[v_m-x(t+t_m)]\equiv 1$ when $v_m\to\infty$,

$$\begin{aligned}z(t)&=\prod_{m\in S} u[v_m-x(t+t_m)]\\ &=\prod_{m\in\bar{S}_0} u[v_m-x(t+t_m)],\end{aligned}$$

when $v_m \to \infty$ for $m \in S_0$ and, therefore, the result follows immediately. □

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