Comparison of Autocorrelation and Cross-Correlation Methods for Signal-Selective TDOA Estimation

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Abstract—Two recently proposed algorithms for signal-selective time-difference-of-arrival estimation are compared in terms of their implementation and their mean-squared errors (MSE's). A tradeoff between ease of implementation and MSE performance is shown to exist. It is shown that the MSE is independent of the distance between sensors for both algorithms.

The purpose of this correspondence is to clarify the relative advantages of two recently proposed signal-selective algorithms for time-difference-of-arrival (TDOA) estimation. It is explained that the algorithm called spectral coherence alignment (SPECCOA) or spectral correlation product (SPECCORP) [1]–[3], which uses cyclic cross-correlation measurements and requires a search over the TDOA parameter, estimates TDOA directly. It is also explained that the algorithm called cyclic phase difference (CPD)[1] [3], which uses cyclic autocorrelation measurements and no search, estimates the phase difference of regenerated spectral lines and this phase-difference estimate can be converted to a TDOA estimate without ambiguity only if the magnitude of the actual TDOA does not exceed half the reciprocal of the cycle frequency. This ambiguity problem severely limits the allowable distance between sensors unless other types of measurements can resolve the ambiguity. It is shown that although the mean-squared error (MSE) of each algorithm is independent of the distance between sensors, the MSE of the CPD algorithm is substantially larger than that of the SPEC-COA algorithm.

The cyclic crosscorrelation of two signals $x(t)$ and $y(t)$ is defined by [5]

$$R_{yx}^{\alpha}(\tau) \triangleq \langle y(t + \tau/2)x^*(t - \tau/2)e^{-i2\pi\alpha t} \rangle, \qquad (1)$$

where $\langle \cdot \rangle$ denotes average over time t, and α is the cycle frequency parameter. Cycle frequencies of interest for communication and telemetry signals include, for example, keying rates and their harmonics, doubled carrier frequencies, and sums and differences of these [2], [5]. (For analytic signals, the conjugate is to be omitted from x in (1) when α involves the doubled carrier frequency.)

[1]The CPD algorithm was originally proposed by Gardner [3] but was independently proposed by Xu and Kailath [4].

For the signal model

$$x(t) = s(t) + n(t)$$

$$y(t) = s(t - D) + m(t) \qquad (2)$$

where $s(t)$ is a signal of interest that exhibits cyclostationarity with a particular cycle frequency α (i.e., $R_{ss}^{\alpha}(\tau) \neq 0$), D is the TDOA, and $n(t)$ and $m(t)$ are interfering signals and noise, which do not exhibit cyclostationarity with the particular cycle frequency α, $(R_n^{\alpha}(\tau) \equiv R_m^{\alpha}(\tau) \equiv R_{sn}^{\alpha}(\tau) \equiv R_{sm}^{\alpha}(\tau) \equiv 0)$, we have

$$R_{yx}^{\alpha}(\tau) = R_s^{\alpha}(\tau - D) e^{-i\pi\alpha D} \qquad (3)$$

$$R_x^{\alpha}(\tau) = R_s^{\alpha}(\tau) \qquad (4)$$

$$R_y^{\alpha}(\tau) = R_s^{\alpha}(\tau) e^{-i2\pi\alpha D}. \qquad (5)$$

By doing a least squares fit, with respect to the TDOA parameter $\tilde{D}$, of an estimate of (3) to an estimate of (4)

$$\min_{\tilde{D}} \int |\hat{R}_{yx}^{\alpha}(\tau) - \hat{R}_x^{\alpha}(\tau - \tilde{D}) e^{-i\pi\alpha\tilde{D}}|^2 \, d\tau \qquad (6)$$

we obtain the SPECCOA algorithm

$$\hat{D} = \arg \max_{\tilde{D}} \text{Re} \left\{ \int \hat{R}_{yx}^{\alpha}(\tau) \hat{R}_x^{\alpha}(\tau - \tilde{D})^* e^{i\pi\alpha\tilde{D}} \, d\tau \right\}. \qquad (7)$$

Using Parseval's relation for the cyclic cross spectrum

$$S_{yx}^{\alpha}(f) = \int_{-\infty}^{\infty} R_{yx}^{\alpha}(\tau) e^{-i2\pi f\tau} \, d\tau \qquad (8)$$

(7) can be reexpressed in the frequency domain as

$$\hat{D} = \arg \max_{\tilde{D}} \text{Re} \left\{ \int \hat{S}_{yx}^{\alpha}(f) \hat{S}_x^{\alpha}(f)^* e^{i2\pi(f + \alpha/2)\tilde{D}} \, df \right\} \qquad (9)$$

which can be implemented efficiently using the FFT algorithm [7] (e.g., using the frequency-smoothed cyclic periodogram method [5]).

Similarly, by doing a least squares fit of an estimate of (5) to an estimate of (4)

$$\min_{\tilde{D}} \int |\hat{R}_y^{\alpha}(\tau) - \hat{R}_x^{\alpha}(\tau) e^{-i2\pi\alpha\tilde{D}}|^2 \, d\tau \qquad (10)$$

we obtain the CPD algorithm

$$\hat{D} = \frac{1}{2\pi\alpha} \text{angle} \left\{ \int \hat{R}_y^{\alpha}(\tau) \hat{R}_x^{\alpha}(\tau)^* \, d\tau \right\} \qquad (11)$$

or

$$\hat{D} = \frac{1}{2\pi\alpha} \text{angle} \left\{ \int \hat{S}_y^{\alpha}(f) \hat{S}_x^{\alpha}(f)^* \, df \right\} \qquad (12)$$

which also can be implemented efficiently using an FFT algorithm [5], [7]. If the angle in (11) or (12) exceeds $\pm\pi$, then CPD will produce an ambiguous estimate (in other words, $\hat{D}$ is given modulo $1/\alpha$).

Using a binary phase-shift-keyed signal $s(t)$, with rectangular keying envelope, in additive white Gaussian noise $n(t)$ and $m(t)$ (where $s(t)$, $n(t)$, $m(t)$ are all independent) with SNR = 0 dB and collection bandwidth B equal to eight times the keying rate, the two algorithms (9) and (12) were simulated using data collection times ranging from $T = 1024T_s$ to $T = 8192T_s$, where $T_s = 1/2B$ is the time sampling increment. One thousand Monte Carlo trials were used to estimate the MSE for each algorithm, and the results

TABLE I
RMSE FOR SPECCOA

TDOA, D	Collection Time, T/T_s			
	1.024	2.048	4.096	8.192
0.375	0.076	0.055	0.030	0.014
3.375	0.074	0.052	0.028	0.008
33.375	0.079	0.053	0.029	0.013

are reported in Tables I and II. For the case where $T = 1024T_s$, there were two outliers among the 1000 trials and these were discarded. There were no outliers for $T > 1024T_s$. Since the smallest TDOA, $D = 0.375T_s$, in Table I is only a fraction of the sampling increment and since the other values of TDOA contain a fractional part of T_s, the estimates of the cyclic spectral densities in (9) were zero padded in f before the inverse FFT indicated in (9) was performed. A zero padding factor of 8 was used. The cyclic spectral densities in (9) and (12) were estimated using an FFT in the frequency-smoothed cyclic periodogram method [5] with a smoothing-window width of one tenth the keying rate.

It can be seen from Tables I and II that the MSE performance of CPD is inferior to that of SPECCOA. This and the ambiguity problem are the prices paid for the simpler implementation of CPD for which no search over the TDOA parameter is needed and no transmission of raw data from one sensor to another (or from both sensors to a central processing station) is needed since no *cross* correlations are used. It should be clarified that although the MSE of the TDOA is essentially independent of the distance L between sensors, the MSE in angle of arrival (AOA) is strongly dependent on this distance. For example, for sensors in the far field of the source (planar wavefronts), the AOA is given by

$$\theta = \sin^{-1} \left(\frac{cD}{L} \right) \qquad (13)$$

and, therefore, for small MSE in D, we have

$$\text{MSE}_{\theta} \cong \left| \frac{L}{c} \cos \theta \right|^{-2} \text{MSE}_D \qquad (14)$$

where c is the speed of propagation. Also, it should be explained that the resolution capability of TDOA estimation algorithms is normally strongly dependent on the distance between sensors, but for the signal-selective algorithms discussed here, TDOA resolution is not a problem except in the relatively rare event that more than one signal with the same cycle frequency α used by the algorithm is received. In this case, CPD is inappropriate but SPECCOA will function properly if the separation between adjacent TDOAs exceeds the reciprocal of the sum of the two corresponding signals' bandwidths. Even though SPECCOA contains a search over the unknown TDOA parameter $\tilde{D}$, whereas the CPD does not, the two algorithms exhibit roughly the same computational complexity. Given that CPD has a serious ambiguity problem, we can compare it to a modified SPECCOA algorithm. This modified SPECCOA searches only over the unambiguous $\tilde{D}$ for the CPD: $\tilde{D} \in (0, 1/\alpha)$. For the simulations reported herein, the modified SPECCOA algorithm would search over 128 points, whereas the unmodified SPECCOA algorithm would search over $8T/T_s$ points.

We point out that the MSE for SPECCOA can approach zero for the simulated cases because the TDOAs lie on the search point $(kT_s/8)$. However, the MSE for SPECCOA is small even when the TDOA lies between search points. In this case, the minimum achievable MSE for SPECCOA is equal to the square of the separation between the TDOA and the nearest search point. For ex-

TABLE II
RMSE FOR CPD

TDOA, D	Collection Time, T/T_s			
	1,024	2,048	4,096	8,192
0.375	0.46	0.30	0.22	0.16
3.375	0.57	0.36	0.24	0.17
33.375	0.58	0.36	0.25	0.17

ample, for the collect time $T = 2048T_s$ in the experiments above, and with $D = 0.375T_s + T_s/16 = 0.4375T_s$, we find that the measured root MSE is equal to 0.065.

As a final remark, it is mentioned that, unlike CPD, SPECCOA has optimality properties. That is, it is a nonparamteric implementation of a maximum-SNR cross-sensor quadratic spectral line regenerator for a cyclostationary signal in additive white Gaussian noise, which is a component of the weak-signal maximum-likelihood joint detector and TDOA estimator [6]. Thus, it is not surprising that SPECCOA outperforms CPD. The performance of SPECCOA has been evaluated in a variety of signal, noise, and interference environments in [7].

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