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Degrees of cyclostationarity and their application to signal detection and estimation

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Abstract. The problem of defining an appropriate measure of the degree of nonstationarity for stochastic processes that exhibit cyclostationarity is addressed. After discussing several candidate measures of degree of nonstationarity, one particularly promising measure is adopted. By decomposing this measure, several component measures are arrived at. Bounds on these measures are derived and their utility in applications involving signal detection and estimation is established. Examples are presented to illustrate the calculation of degrees of nonstationarity for several types of cyclostationary signals.

Zusammenfassung. Die vorliegende Veröffentlichung behandelt die Definierung der entsprechende Maßgabe von Nichtstationäritätsgrad des stochastischen Prozeßen mit zyklisch-stationäre Eigenschaften. Nach einer Überlegung über die verschiedenen Maßgaben bei Nichtstationäritätsgrad wurde eine besonders versprechende Maßgabe angenommen. Es wird gezeigt, daß die einige wesentliche Maßgaben mit Hilfe der Dekomposition erreicht werden können. Zusätzlich wurden die Grenzen dieser Maßgabe hergeleitet und ihre Wirksamkeit wurde in Anwendungen enthaltend die Signaldetektion und Schätzung festgesetzt. Als Anwendungsbeispiel wird die Berechnung der Nichtstationäritätsgrad für einige der kennzeichender zyklisch-stationäre Signale dargestellt.

Résumé. Le problème de la définition d'une mesure appropriée du degré de nonstationnarité des processus stochastiques produisant une cyclostationnarité est présenté. Après avoir discuté les différentes mesures étudiées, une mesure particulièrement prometteuse est adoptée. En décomposant cette mesure on a obtenu plusieures composantes. Le domaine de ces mesures est dérivé et l'utilité de leur application dans la détection et l'estimation des signaux est définie. Quelques exemples sont présentés pour illustrer le calcul des degrés de la nonstationnarité de plusieurs types de signaux cyclostationnaires.

Keywords. Cyclostationary process, degree of cyclostationarity, instantaneous spectrum, spectral correlation.

1. Introduction

The purpose of this paper is to present an approach to defining and analyzing the degree of nonstationarity for a particular class of nonstationary stochastic processes. The class of interest consists of periodic or multiply-periodic (almost periodic [2]) wide-sense nonstationary processes, that is, processes exhibiting periodic or multiply-periodic nonstationarity of the autocorrelation. This type of nonstationarity is commonly referred to as wide-sense cyclostationarity.

The problem of defining an appropriate measure of degree of cyclostationarity (DCS) was first posed in [9] in connection with application to linear minimum mean-squared error waveform

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estimation. As explained in [9], there is need for a measure of DCS that is both convenient to evaluate and useful in terms of applications.

Our subsequent work has shown that there is no unique definition of DCS that is most appropriate for all applications. For example, the most appropriate definition for application to the detection of cyclostationary signals in noise is not necessarily the same as the most appropriate definition for application to the estimation of cyclostationary signals in noise. (This is explained in Section 4.)

The concept of the distance to the nearest stationary process is, at first glance, an attractive way to define DCS, but it is not clear if the technical problems associated with this approach can be overcome. For example, an appropriate metric is the square root of the time-averaged mean-squared error. But evaluation of this metric requires a joint probabilistic model, viz., a crosscorrelation function, for all processes in the metric space. How can such joint properties be a part of the definition of a universal metric space, e.g., a metric space containing *all* jointly wide-sense cyclostationary processes? Also, it is not clear whether the closest stationary process to a particular cyclostationary process even exists.

An alternative is to simply use the distance between the particular cyclostationary process of interest and a stationarized (by phase randomization [3]) version of it. But this distance turns out to depend on nothing more than the stationarized (time-averaged) autocorrelation of the cyclostationary process and the probability distribution of the stationarizing random phase, which depends on the cyclostationarity of the autocorrelation in only a trivial way (i.e., it depends on only the period(s)).

We have found a more workable approach to be that based on the concept of the distance between the nonstationary (instantaneous) autocorrelation and the closest stationary autocorrelation or, equivalently, the distance between the nonstationary (instantaneous) spectrum and the closest stationary spectrum. In fact, the closest stationary autocorrelation (or spectrum) turns out to be identical to the autocorrelation (or spectrum) of the stationarized version of the cyclostationary process (i.e., that obtained by phase randomization).

In this paper, we pursue this approach to defining DCS, and we show how this leads to several distinct component-measures of DCS that are useful in applications to signal detection and estimation.

2. Background and definition of DCS

For a wide-sense nonstationary stochastic process $X(t)$, the instantaneous probabilistic autocorrelation is defined to be the expected value of the symmetrized lag product,

$$R_X(t,\tau) \triangleq \mathrm{E}\{X(t+\tau/2)X^*(t-\tau/2)\} \tag{1}$$

(where X^* denotes the complex conjugate of X), and the instantaneous probabilistic spectrum is defined to be the Fourier transform, in the lag parameter, of the instantaneous probabilistic autocorrelation,

$$S_X(t,f) \triangleq \int_{-\infty}^{\infty} R_X(t,\tau)\,\mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}\tau. \tag{2}$$

For a wide-sense stationary stochastic process, $R_X(t,\tau)$ and, therefore, $S_X(t,f)$ are independent of the time-location parameter t,

$$R_X(t,\tau) = R_X(\tau), \tag{3}$$

$$S_X(t,f) = S_X(f). \tag{4}$$

Furthermore, in this case $S_X(t,f) = S_X(f)$ has the special interpretation of being a spectral density function. That is, it can be shown [7] that

$$S_X(f) = \lim_{T\to\infty} \mathrm{E}\left\{\frac{1}{T}|\tilde{X}_T(t,f)|^2\right\}, \tag{5}$$

where

$$\tilde{X}_T(t,f) = \int_{t-T/2}^{t+T/2} X(u)\,\mathrm{e}^{-\mathrm{i}2\pi fu}\,\mathrm{d}u. \tag{6}$$

Thus, since $|\tilde{X}_T(t,f)|$ represents the strength of

spectral components of $X(\cdot)$ in a band of approximate width $1/T$, centered at frequency f, then $S_X(f)$ is the spectral density of mean square, or, of expected power, and is typically called the power spectral density function.

For a generally nonstationary process $X(t)$, $S_X(t,f)$ does not have any similar interpretation as some type of spectral density function. However, if the nonstationarity is due only to periodic or multiply-periodic time variation, then $S_X(t,f)$ does have a special interpretation. To see this, we expand the multiply-periodic (almost periodic) function $S_X(\cdot,f)$ in a Fourier series

$$S_X(t,f)=\sum_{\alpha} S_X^{\alpha}(f)\,\mathrm{e}^{\mathrm{i}2\pi\alpha t}, \tag{7}$$

with Fourier coefficients

$$S_X^{\alpha}(f)\triangleq\langle S_X(t,f)\,\mathrm{e}^{-\mathrm{i}2\pi\alpha t}\rangle = \lim_{Z\to\infty}\frac{1}{Z}\int_{-Z/2}^{Z/2} S_X(t,f)\,\mathrm{e}^{-\mathrm{i}2\pi\alpha t}\,\mathrm{d}t, \tag{8}$$

where α, which is called the cycle frequency parameter, ranges over all values for which $S_X^{\alpha}(f)\not\equiv 0$. It can be shown that the Fourier coefficient function $S_X^{\alpha}(f)$ is given by [7]

$$S_X^{\alpha}(f)=\lim_{T\to\infty}\left\langle \mathrm{E}\left\{\frac{1}{T}\tilde{X}_T(t,f+\alpha/2)\times\tilde{X}_T^{*}(t,f-\alpha/2)\right\}\right\rangle. \tag{9}$$

Thus, $S_X^{\alpha}(f)$ is the spectral density of correlation between spectral components located at frequencies $f+\alpha/2$ and $f-\alpha/2$, and is called the spectral correlation density function. It follows from (1), (2) and (8) that $S_X^{\alpha}(f)$ can also be characterized as the Fourier transform of the Fourier-coefficient function $R_X^{\alpha}(\tau)$ in the Fourier series expansion of the instantaneous autocorrelation

$$R_X(t,\tau)=\sum_{\alpha} R_X^{\alpha}(\tau)\,\mathrm{e}^{\mathrm{i}2\pi\alpha t}, \tag{10}$$

where

$$R_X^{\alpha}(\tau)\triangleq\langle R_X(t,\tau)\,\mathrm{e}^{-\mathrm{i}2\pi\alpha t}\rangle. \tag{11}$$

That is,

$$S_X^{\alpha}(f)=\int_{-\infty}^{\infty} R_X^{\alpha}(\tau)\,\mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}\tau. \tag{12}$$

The average value of the instantaneous spectrum is simply the power spectral density function when the process is stationary:

$$S_X^{0}(f)\triangleq\langle S_X(t,f)\rangle=S_X(t,f)=S_X(f). \tag{13}$$

Similarly, the average value of the instantaneous autocorrelation is simply the conventional autocorrelation when the process is stationary,

$$R_X^{0}(\tau)\triangleq\langle R_X(t,\tau)\rangle=R_X(t,\tau)=R_X(\tau), \tag{14}$$

and we have, as a special case of (12), corresponding to $\alpha=0$,

$$S_X(f)=\int_{-\infty}^{\infty} R_X(\tau)\,\mathrm{e}^{-\mathrm{i}2\pi f\tau}\,\mathrm{d}\tau, \tag{15}$$

which, together with (5), is known as the Wiener–Khinchin relation. By analogy, (12), together with (9), is called the cyclic Wiener–Khinchin relation [7]. Because of the analogy between (15) (and (5)) and (12) (and (9)), $R_X^{\alpha}(\tau)$ is called the cyclic autocorrelation and $S_X^{\alpha}(f)$ is called the cyclic spectral density [7]. Any process for which $R_X^{\alpha}(\tau)\not\equiv 0$ for some $\alpha\neq 0$ is said to exhibit cyclostationarity.

As a measure of the DCS of a process $X(t)$ that exhibits cyclostationarity, we take the normalized minimum time-averaged lag-integrated squared difference between the cyclostationary autocorrelation $R_X(t,\tau)$ and the closest stationary autocorrelation $R(\tau)$:

$$\mathrm{DCS}\triangleq\frac{\min\limits_{R(\cdot)}\displaystyle\int_{-\infty}^{\infty}\langle|R_X(t,\tau)-R(\tau)|^2\rangle\,\mathrm{d}\tau}{\displaystyle\int_{-\infty}^{\infty}|\langle R_X(t,\tau)\rangle|^2\,\mathrm{d}\tau}. \tag{16}$$

It can be shown that the closest stationary autocorrelation is simply the time-averaged cyclostationary autocorrelation

$$R(\tau)=\langle R_X(t,\tau)\rangle\triangleq R_X^{0}(\tau). \tag{17}$$

Furthermore, this is also identical to the autocorrelation of the stationarized version of $X(t)$ obtained by phase randomization [3].

By using Parseval's relation (see Section 3) it can be seen that the DCS (16) is identical to the following DCS defined in terms of the instantaneous spectrum:

$$\mathrm{DCS} \triangleq \frac{\min\limits_{S(\cdot)} \int_{-\infty}^{\infty} \langle |S_X(t,f) - S(f)|^2 \rangle \, \mathrm{d}f}{\int_{-\infty}^{\infty} |\langle S_X(t,f) \rangle|^2 \, \mathrm{d}f}, \tag{18}$$

and the closest stationary spectrum is simply the Fourier transform of the closest stationary autocorrelation (17):

$$S(f) = \langle S_X(t,f) \rangle = S_X^0(f). \tag{19}$$

For all processes, DCS $\geqslant 0$. For a stationary process, it follows from (13) or (14) that DCS $= 0$. Also, for a nonstationary process, it can be seen that if the nonstationarity is only transient so that the process is asymptotically stationary ($S_X(t,f) \to S_X^0(f)$ as $t \to \infty$) then DCS $= 0$. But if the process exhibits cyclostationarity then DCS > 0.

The objective of this paper is to introduce various decompositions of DCS that can be directly related to the performances of signal processors (detectors and estimators) that optimally exploit cyclostationarity. However, before proceeding, it is explained that all of the foregoing definitions, as well as the results presented in the following sections, can be given nonprobabilistic interpretations involving only time averages rather than expected values.

Specifically, for a time-series $x(t)$, such as one sample path of a stochastic process $X(t)$, the nonprobabilistic counterpart of (9) (and, therefore, (5)) is

$$\hat{S}_x^\alpha(f) \triangleq \lim_{T \to \infty} \left\langle \frac{1}{T} \tilde{x}_T(t, f + \alpha/2) \tilde{x}_T^*(t, f - \alpha/2) \right\rangle, \tag{20}$$

where $\tilde{x}_T$ stands for the short-time Fourier transform of $x(t)$ analogous to (6) and, similarly, the nonprobabilistic counterpart of (11) (with (1) substituted in) is

$$\hat{R}_x^\alpha(\tau) \triangleq \left\langle \frac{1}{T} x(t + \tau/2) x^*(t - \tau/2) \, \mathrm{e}^{-\mathrm{i}2\pi\alpha t} \right\rangle. \tag{21}$$

And, it can be shown that (20) and (21) are a Fourier transform pair [5]

$$\hat{S}_x^\alpha(f) = \int_{-\infty}^{\infty} \hat{R}_x^\alpha(\tau) \, \mathrm{e}^{-\mathrm{i}2\pi f\tau} \, \mathrm{d}\tau, \tag{22}$$

analogous to (12). Relation (22) is called the cyclic Wiener relation [5]. In terms of definitions (20) and (21), we can, by analogy with (7) and (10), define the nonprobabilistic instantaneous spectrum

$$\hat{S}_x(t,f)) \triangleq \sum_\alpha \hat{S}_x^\alpha(f) \, \mathrm{e}^{\mathrm{i}2\pi\alpha t}, \tag{23}$$

and the nonprobabilistic instantaneous autocorrelation

$$\hat{R}_x(t,\tau) \triangleq \sum_\alpha \hat{R}_x^\alpha(\tau) \, \mathrm{e}^{\mathrm{i}2\pi\alpha t}, \tag{24}$$

which are a Fourier transform pair analogous to (2). It follows from (20)–(24) that the nonprobabilistic counterpart of the DCS (18), (19) is simply

$$\mathrm{DCS} \triangleq \frac{\int_{-\infty}^{\infty} [\langle |\hat{S}_x(t,f)|^2 \rangle - |\langle \hat{S}_x(t,f) \rangle|^2] \, \mathrm{d}f}{\int_{-\infty}^{\infty} |\langle \hat{S}_x(t,f) \rangle|^2 \, \mathrm{d}f} \tag{25}$$

and in a similar manner we obtain the nonprobabilistic counterpart of (16).

3. Decompositions of DCS

It follows directly from the Fourier transform and Fourier series properties applied to (2) and (7) that we have the following Parseval's relations [5, 10]:

$$\langle |S_X(t,f)|^2 \rangle = \sum_\alpha |S_X^\alpha(f)|^2, \tag{26}$$

$$\langle |R_X(t,\tau)|^2 \rangle = \sum_\alpha |R_X^\alpha(\tau)|^2, \tag{27}$$

$$\int_{-\infty}^{\infty} |S_X(t,f)|^2 \, \mathrm{d}f = \int_{-\infty}^{\infty} |R_X(t,\tau)|^2 \, \mathrm{d}\tau, \tag{28}$$

$$\int_{-\infty}^{\infty} |S_X^\alpha(f)|^2 \, \mathrm{d}f = \int_{-\infty}^{\infty} |R_X^\alpha(\tau)|^2 \, \mathrm{d}\tau. \tag{29}$$

We also have the definitions

$$\langle S_X(t,f) \rangle = S_X^0(f), \tag{30}$$

$$\langle R_X(t,\tau) \rangle = R_X^0(\tau). \tag{31}$$

Substituting (26)-(31) into the definition (18), (19) of DCS yields the alternative expressions

$$\mathrm{DCS} = \frac{\sum_{\alpha \neq 0} \int_{-\infty}^{\infty} |S_X^\alpha(f)|^2 \, \mathrm{d}f}{\int_{-\infty}^{\infty} |S_X^0(f)|^2 \, \mathrm{d}f} \tag{32}$$

$$= \frac{\sum_{\alpha \neq 0} \int_{-\infty}^{\infty} |R_X^\alpha(\tau)|^2 \, \mathrm{d}\tau}{\int_{-\infty}^{\infty} |R_X^0(\tau)|^2 \, \mathrm{d}\tau}. \tag{33}$$

These alternative expressions suggest the cycle-frequency-decomposed measure of DCS

$$\mathrm{DCS}_f^\alpha \triangleq \frac{\int_{-\infty}^{\infty} |S_X^\alpha(f)|^2 \, \mathrm{d}f}{\int_{-\infty}^{\infty} |S_X^0(f)|^2 \, \mathrm{d}f} = \frac{\int_{-\infty}^{\infty} |R_X^\alpha(\tau)|^2 \, \mathrm{d}\tau}{\int_{-\infty}^{\infty} |R_X^0(\tau)|^2 \, \mathrm{d}\tau}, \tag{34}$$

which in turn can be expressed in terms of the following frequency-decomposed and time-decomposed measures:

$$\mathrm{DCS}_f^\alpha \triangleq \frac{|S_X^\alpha(f)|^2}{S_X^0(f+\alpha/2) S_X^0(f-\alpha/2)}, \tag{35}$$

$$\mathrm{DCS}_\tau^\alpha \triangleq \frac{|R_X^\alpha(\tau)|^2}{|R_X^0(0)|^2}. \tag{36}$$

These two latter measures are both magnitude-squared correlation coefficients. That is,

$$\mathrm{DCS}_\tau^\alpha = |\rho_\tau^\alpha|^2 \tag{37}$$

and

$$\mathrm{DCS}_f^\alpha = |\rho_f^\alpha|^2, \tag{38}$$

where

$$\rho_\tau^\alpha \triangleq \frac{R_{UV}}{\sqrt{R_{UU} R_{VV}}}, \tag{39}$$

$$\rho_f^\alpha \triangleq \lim_{T \to \infty} \frac{R_{P_T Q_T}}{\sqrt{R_{P_T P_T} R_{Q_T Q_T}}}, \tag{40}$$

and where

$$R_{UV} \triangleq \langle \mathrm{E}\{U(t) V^*(t)\} \rangle, \tag{41}$$

with

$$U(t) \triangleq X(t+\tau/2) \, \mathrm{e}^{-\mathrm{i}\pi\alpha t}, \quad V(t) \triangleq X(t-\tau/2) \, \mathrm{e}^{+\mathrm{i}\pi\alpha t} \tag{42}$$

and

$$P_T(t) \triangleq \tilde{X}_T(t, f+\alpha/2), \quad Q_T(t) \triangleq \tilde{X}_T(t, f-\alpha/2). \tag{43}$$

It follows from (32) and (34) that DCS can be composed as follows:

$$\mathrm{DCS} = \sum_{\alpha \neq 0} \mathrm{DCS}^\alpha, \tag{44}$$

and it follows from (34)-(36) that DCS^α can be composed as follows:

$$\mathrm{DCS}^\alpha = \frac{\int_{-\infty}^{\infty} \mathrm{DCS}_\tau^\alpha \, \mathrm{d}\tau}{\int_{-\infty}^{\infty} |\rho_\tau^0|^2 \, \mathrm{d}\tau} \tag{45}$$

and

$$\mathrm{DCS}^\alpha = \frac{\int_{-\infty}^{\infty} \mathrm{DCS}_f^\alpha S_X^0(f+\alpha/2) S_X^0(f-\alpha/2) \, \mathrm{d}f}{\int_{-\infty}^{\infty} |S_X^0(f)|^2 \, \mathrm{d}f}. \tag{46}$$

In the next section DCS and its components DCS^α, DCS_τ^α and DCS_f^α are upper- and lower-

bounded and then related to the performances of signal processors that optimally exploit cyclostationarity.

4. DCS and performance

We begin with some general bounds on DCS and its components. It is clear from (34)-(36) and (44) that

$$\mathrm{DCS} \geqslant \max \mathrm{DCS}^{\alpha} \geqslant 0, \tag{47}$$

and that

$$\mathrm{DCS}_f^{\alpha} \geqslant 0 \tag{48}$$

and

$$\mathrm{DCS}_{\tau}^{\alpha} \geqslant 0. \tag{49}$$

Thus DCS and all its components are nonnegative. Furthermore, all the components of DCS are upper-bounded by unity, which can be seen as follows.

Using the inequality [7, 5]

$$|S_X^{\alpha}(f)|^2 \leqslant S_X^0(f+\alpha/2)S_X^0(f-\alpha/2) \tag{50}$$

and the Cauchy-Schwarz inequality, it follows from (34) that

$$\mathrm{DCS}^{\alpha} \leqslant 1. \tag{51}$$

In addition, if $X(t)$ has bandwidth B, i.e., if

$$S_X^0(f) = 0, \quad |f| \geqslant B, \tag{52}$$

then using (50) in (34) results in

$$\mathrm{DCS}^{\alpha} \leqslant 1 - |\alpha|/2B, \tag{53}$$

where $|\alpha| < 2B$. Also, because of (50), we have

$$\mathrm{DCS}_f^{\alpha} \leqslant 1. \tag{54}$$

Similarly, it can be shown that

$$\mathrm{DCS}_{\tau}^{\alpha} \leqslant 1. \tag{55}$$

Inequalities (54) (or (50)) and (55) are direct results of the fact that DCS_f^{α} and $\mathrm{DCS}_{\tau}^{\alpha}$ are both magnitude-squared correlation coefficients.

DCS itself can also be upper bounded when $X(t)$ is bandlimited. That is, if $X(t)$ is real and has only L positive cycle frequencies (and, therefore, L negative cycle frequencies), then it follows from (44) and (51) that

$$\mathrm{DCS} \leqslant 2L. \tag{56}$$

In addition, if $X(t)$ exhibits cyclostationarity with only one period T_0, and $X(t)$ is bandlimited as in (52), then $L \leqslant 2BT_0$. Also, in this case, it follows from (44) and (53) that

$$\begin{aligned}\mathrm{DCS} &\leqslant \sum_{k=-L}^{L} (1-|k|/2BT_0) - 1 \\ &= 2L - L(L+1)/2BT_0,\end{aligned} \tag{57}$$

and this upper bound never goes below $L-1$.

In the remainder of this section, we relate DCS and its components to the performances of signal detectors and estimators that optimally exploit cyclostationarity.

Beginning with the most elementary relation, we consider the problem of estimating $U(t)$ in (42) using a linear transformation, $\hat{U}(t) = hV(t)$, of $V(t)$ in (42). It can be shown that the normalized minimum (with respect to h) value of the time-averaged mean-squared error

$$e_{\tau}^{\alpha} = \langle \mathrm{E}\{[\hat{U}(t) - U(t)]^2\} \rangle \tag{58}$$

is given by

$$\frac{\min\{e_{\tau}^{\alpha}\}}{\langle \mathrm{E}\{|U(t)|^2\}\rangle} = 1 - \mathrm{DCS}_{\tau}^{\alpha}. \tag{59}$$

Thus, estimation is possible if and only if $X(t)$ exhibits cyclostationarity with cycle frequency α, and the accuracy of the estimation is determined directly by $\mathrm{DCS}_{\tau}^{\alpha}$. This type of estimation plays a fundamental role in blind-adaptive spatial filtering of signals that exhibit cyclostationarity [1, 7].

Let us now consider the problem of estimating the process $X(t)$ throughout a spectral band of width B centered at frequency $f_0 + \alpha/2$—call this bandlimited part of $X(t)$ $X_+(t)$—using a linearly filtered version, $\hat{X}_+(t) = h(t) \otimes X_-(t)$ (where $\otimes$ denotes convolution), of $X_-(t)$, which is the portion of $X(t)$ in the spectral band of width B

centered at $f_0-\alpha/2$, but shifted in frequency to the band of width B centered at $f_0+\alpha/2$. It follows from the optimum filtering formulas in [7] or [5] that the time-averaged minimum (with respect to $h(\cdot)$) value of the mean-squared error

$$e_f^\alpha = \langle \mathrm{E}\{[\hat{X}_+(t)-X_+(t)]^2\}\rangle \tag{60}$$

is given by

$$\min\{e_f^\alpha\} = \int_{f_0-B/2}^{f_0+B/2} S_X(f+\alpha/2)[1-\mathrm{DCS}_f^\alpha]\,\mathrm{d}f. \tag{61}$$

Thus, estimation is possible if and only if $X(t)$ exhibits cyclostationarity with cycle frequency α, and the accuracy of estimation is determined directly by DCS_f^α. This type of estimation, which exploits the inherent frequency diversity of cyclostationary processes (a result of the correlation among spectral components in disjoint bands), can be useful for mitigating the effects of frequency-selective fading (e.g., due to multipath propagation) and/or cochannel interference [5, 7, 8].

A different type of problem from the two preceding estimation problems involves the detection of the presence of a signal $X(t)$ buried in noise. A useful performance measure for weak random-signal detection is a type of detector-output SNR called deflection and defined by

$$d \triangleq \frac{|\mathrm{mean}\{Y(t)|\mathrm{yes}\}-\mathrm{mean}\{Y(t)|\mathrm{no}\}|^2}{\mathrm{var}\{Y(t)|\mathrm{no}\}}, \tag{62}$$

where $Y(t)$ is the output of the detector and $|$yes and $|$no indicate conditioning on the presence and absence of the signal at the input to the detector.

It can be shown that, with respect to all quadratic detectors of the form

$$Y(t) = \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} k(u,v)\,W(t+u) \times W(t+v)\,\mathrm{d}u\,\mathrm{d}v, \tag{63}$$

where the kernel $k(\cdot,\cdot)$ determines the particular detector, and $W(t)$ is either signal plus noise, $X(t)+N(t)$, or just noise, $N(t)$, the maximum attainable value of deflection is given by [5–7]

$$\max\{d\} = \frac{T}{2N_0^2}\sum_\alpha \int_{-\infty}^{\infty} |S_X^\alpha(f)|^2\,\mathrm{d}f, \tag{64}$$

where T stands for the collect time of the detector, and N_0 is the spectral density of the Gaussian noise (i.e., $S_N^0(f)=N_0$).

On the other hand, if the cyclostationarity of the signal is ignored (by using a random phase to stationarize the signal model [3, 7]), then the maximum attainable deflection is only

$$\max\{d^0\} = \frac{T}{2N_0^2}\int_{-\infty}^{\infty} |S_X^0(f)|^2\,\mathrm{d}f. \tag{65}$$

It follows from (64), (65) and (34) that the gain in deflection that is attainable by exploiting cyclostationarity is given by

$$\frac{\max\{d\}}{\max\{d^0\}} = \mathrm{DCS}+1. \tag{66}$$

It can be shown that the optimum quadratic detector that exploits cyclostationarity generates maximum-SNR spectral lines at all the cycle frequencies α of the signal $X(t)$, and then adds the noisy Fourier coefficients of the spectral lines together to obtain the detector output $Y(t)$ [4, 7]. It also can be shown that the maximum SNR of each of the regenerated spectral lines is given by [4, 7]

$$\mathrm{SNR}^\alpha = \frac{T}{2N_0^2}\int_{-\infty}^{\infty} |S_X^\alpha(f)|^2\,\mathrm{d}f. \tag{67}$$

When normalized by the output SNR for the detector that ignores cyclostationarity, we obtain the performance ratio

$$\frac{\mathrm{SNR}^\alpha}{\max\{d^0\}} = \mathrm{DCS}^\alpha. \tag{68}$$

Thus, optimum detection performance is directly determined by DCS^α. This type of detection is particularly useful for detecting weak modulated signals, such as direct-sequence spread-spectrum signals, buried in noise [6].

5. Examples

To illustrate the calculation of the various measures of degree of cyclostationarity, and to exemplify the functional forms and numerical values of these measures, we present two examples based on signal models frequently used in communications, telemetry and radar systems.

EXAMPLE 1. As an example of a random process that exhibits cyclostationarity, we consider the amplitude-modulated sine wave

$$X(t) = A(t)\cos(\omega_0 t - \theta), \tag{69}$$

where $A(t)$ is a real stationary process. Using Euler's formula,

$$\cos(\omega_0 t - \theta) = \tfrac{1}{2}\,\mathrm{e}^{\mathrm{i}(\omega_0 t - \theta)} + \tfrac{1}{2}\,\mathrm{e}^{-\mathrm{i}(\omega_0 t - \theta)},$$

we obtain

$$\begin{aligned} &X(t+\tau/2)X^*(t-\tau/2) \\ &\quad = \tfrac{1}{4}A(t+\tau/2)A(t-\tau/2) \\ &\qquad \times[\mathrm{e}^{\mathrm{i}2\omega_0 t}\,\mathrm{e}^{-\mathrm{i}2\theta} + \mathrm{e}^{-\mathrm{i}2\omega_0 t}\,\mathrm{e}^{\mathrm{i}2\theta} + \mathrm{e}^{\mathrm{i}\omega_0\tau} + \mathrm{e}^{-\mathrm{i}\omega_0\tau}]. \end{aligned}$$

Therefore, we have

$$\langle \mathrm{E}\{X(t+\tau/2)X^*(t-\tau/2)\}\,\mathrm{e}^{-\mathrm{i}2\pi\alpha t}\rangle = \begin{cases} \tfrac{1}{4}\langle \mathrm{E}\{A(t+\tau/2)A(t-\tau/2)\}\rangle\,\mathrm{e}^{\mp\mathrm{i}2\theta}, & \alpha = \pm\omega_0/\pi, \\ \tfrac{1}{2}\langle \mathrm{E}\{A(t+\tau/2)A(t-\tau/2)\}\rangle\cos(\omega_0\tau), & \alpha = 0, \\ 0, & \text{otherwise}, \end{cases}$$

because

$$\langle \mathrm{E}\{A(t+\tau/2)A(t-\tau/2)\}\,\mathrm{e}^{\mathrm{i}2\pi\beta t}\rangle \equiv 0$$

for all $\beta \neq 0$ since $A(t)$ is stationary. Thus,

$$R_X^\alpha(\tau) = \begin{cases} \tfrac{1}{4}R_A(\tau)\,\mathrm{e}^{\mp\mathrm{i}2\theta}, & \alpha = \mp\omega_0/\pi, \\ \tfrac{1}{2}R_A(\tau)\cos(\omega_0\tau), & \alpha = 0, \\ 0, & \text{otherwise}. \end{cases} \tag{70}$$

Hence, $R_X^\alpha(\tau)$ is not identically zero for only two nonzero values of α, as shown in Fig. 1. The example shown in Fig. 1 corresponds to an amplitude process $A(t)$ with a triangular autocorrelation.

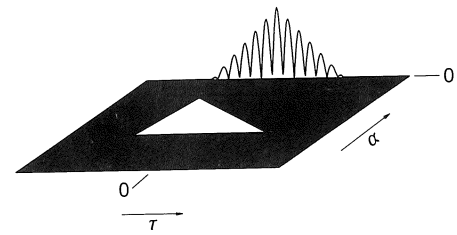

Fig. 1. Graph of function $|R_X^\alpha(\tau)|$ as the height of a surface above the plane with coordinates τ and α for an amplitude modulated sine wave.

It follows from definition (36) and the result (70) that DCS_τ^α for this amplitude modulated process is given by

$$\mathrm{DCS}_\tau^\alpha = \tfrac{1}{4}|\rho_\tau^0|^2, \quad \alpha = \pm\omega_0/\pi, \tag{71}$$

where

$$\rho_\tau^0 = \frac{R_A(\tau)}{R_A(0)} \leq 1. \tag{72}$$

Thus, $\mathrm{DCS}_\tau^\alpha \leq 1/4$ and $\mathrm{DCS}_\tau^\alpha = 1/4$ for $\tau = 0$.

By Fourier transforming (70), we obtain

$$S_X^\alpha(f) = \begin{cases} \tfrac{1}{4}S_A(f)\,\mathrm{e}^{\mp\mathrm{i}2\theta}, & \alpha = \pm\omega_0/\pi, \\ \tfrac{1}{4}S_A(f+\omega_0/2\pi) + \tfrac{1}{4}S_A(f-\omega_0/2\pi), & \alpha = 0, \\ 0, & \text{otherwise}. \end{cases} \tag{73}$$

Thus, only spectral components that are separated by $|\alpha| = \omega_0/\pi$ are correlated. This can be easily understood by expressing the time-series (69) as

$$X(t) = \tfrac{1}{2}A(t)\,\mathrm{e}^{\mathrm{i}\omega_0 t}\,\mathrm{e}^{-\mathrm{i}\theta} + \tfrac{1}{2}A(t)\,\mathrm{e}^{-\mathrm{i}\omega_0 t}\,\mathrm{e}^{\mathrm{i}\theta}.$$

Thus, each spectral component in $A(t)$ is shifted from its original frequency, say f, to $f + \omega_0/2\pi$ and $f - \omega_0/2\pi$. The separation between the frequencies of these pairs of identical (except for a constant phase difference of 2θ) spectral components is ω_0/π. Hence, all such components are completely correlated (as long as $S_A(f) = 0$ for $|f| \geq \omega_0/2\pi$ so that no positively shifted components overlap with

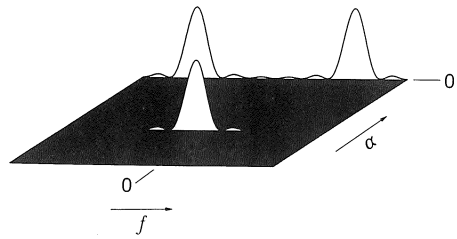

Fig. 2. Graph of function $|S_X^\alpha(f)|$ for an amplitude-modulated sine wave.

any negatively shifted components). A graph of $|S_X^\alpha(f)|$ interpreted as the height of a surface above the plane with coordinates f and α is shown in Fig. 2.

It follows from definition (35) and the result (73) that DCS_f^α for this amplitude modulated process is given by

$$\mathrm{DCS}_f^\alpha = \frac{|S_A(f)|^2}{[S_A(f+\omega_0/\pi)+S_A(f)][S_A(f)+S_A(f-\omega_0/\pi)]} \tag{74}$$

for $\alpha = \pm\omega_0/\pi$. If $S_A(f)=0$ for $|f| \geqslant \omega_0/2\pi$ (which is often the case in practice), then $\mathrm{DCS}_f^\alpha = 1$ for all $|f| < \omega_0/2\pi$.

Also, using definition (34) and the result (70), we find that

$$\mathrm{DCS}^\alpha = \frac{\frac{1}{4}\int_{-\infty}^{\infty} |R_A(\tau)|^2 \,\mathrm{d}\tau}{\int_{-\infty}^{\infty} |R_A(\tau)\cos(\omega_0\tau)|^2 \,\mathrm{d}\tau} \geqslant \frac{1}{4},$$

$$\alpha = \pm\omega_0/\pi, \tag{75}$$

and, using (44), that $\mathrm{DCS} \geqslant \frac{1}{2}$.

EXAMPLE 2. As another example, we consider the real-valued amplitude-modulated pulse train

$$X(t) = \sum_{n=-\infty}^{\infty} A(nT)p(t-nT), \tag{76}$$

where $A(t)$ is a stationary process and $p(t)$ is a non-random finite-energy pulse,

$$\int_{-\infty}^{\infty} p^2(t)\,\mathrm{d}t < \infty.$$

By using the formal characterization

$$X(t) = \left[A(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)\right]\otimes p(t)$$

$$= \left[A(t)\frac{1}{T}\sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi mt/T}\right]\otimes p(t),$$

$$= W(t)\otimes\frac{1}{T}p(t),$$

where

$$W(t) = A(t)\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}2\pi mt/T},$$

we can show that

$$R_X^\alpha(\tau) = \frac{1}{T^2}R_W^\alpha(\tau)\otimes r_p^\alpha(\tau), \tag{77}$$

where

$$r_p^\alpha(\tau) \triangleq \int_{-\infty}^{\infty} p(t+\tau/2)p(t-\tau/2)\,\mathrm{e}^{-\mathrm{i}2\pi\alpha t}\,\mathrm{d}t. \tag{78}$$

By an argument similar to that used in Example 1, we can show that

$$R_W^\alpha(\tau) = R_A(\tau)\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}2\pi m\tau/T}$$

$$= R_A(\tau)T\sum_{n=-\infty}^{\infty}\delta(\tau-nT), \quad \alpha = q/T$$

for all integers q. Therefore, (77) yields

$$R_X^\alpha(\tau) = \begin{cases} \frac{1}{T}\sum_{n=-\infty}^{\infty} R_A(nT)r_p^\alpha(\tau-nT), & \alpha = q/T \\ 0, & \text{otherwise} \end{cases} \tag{79}$$

for all integers q. Thus, $R_X^\alpha(\tau)$ is not identically zero for only values of α that are integer multiples of $1/T$, as shown in Fig. 3. The example shown in Fig. 3 corresponds to a white amplitude sequence $A(nT)$ and a rectangular pulse $p(t)$ of width T.

When the sequence $A(nT)$ is white, then (79) reduces to

$$R_X^\alpha(\tau) = \frac{\sigma_a^2}{T}r_p^\alpha(\tau), \quad \alpha = q/T. \tag{80}$$

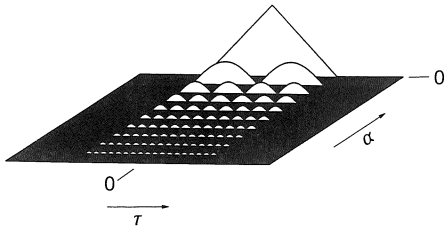

Fig. 3. Graph of the function $|R_X^\alpha(\tau)|$ as the height of a surface above the plane with coordinates τ and α for an amplitude modulated pulse train.

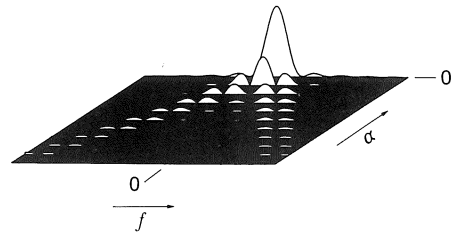

Fig. 4. Graph of the function $|S_X^\alpha(f)|$ for an amplitude-modulated pulse train.

It follows from (36) and (80) that DCS_τ^α for this pulse-amplitude-modulated process is given by

$$\mathrm{DCS}_\tau^\alpha = \left|\frac{r_p^\alpha(\tau)}{r_p^0(\tau)}\right|^2. \tag{81}$$

For example, for a rectangular pulse $p(t)$ of width T, we obtain

$$\mathrm{DCS}_\tau^\alpha = \begin{cases} \left|\dfrac{\sin \pi\alpha(T-|\tau|)}{\pi\alpha T}\right|^2, & |\tau| \leqslant T, \\ 0, & |\tau| > T. \end{cases} \tag{82}$$

For $\alpha = 1/T$, DCS_τ^α peaks at $|\tau| = T/2$ and the peak value is $(1/\pi)^2$. For $\alpha = 2/T$, DCS_τ^α peaks at $|\tau| = T/4$ and $|\tau| = 3T/4$ and the peak value is $(1/2\pi)^2$. Also, using (34) and the result (80) for a rectangular pulse of width T, we obtain for $\alpha = 1/T$

$$\mathrm{DCS}^\alpha = \frac{\displaystyle\int_{-\infty}^{\infty} \left[\frac{\sin \pi(1-|\tau|/T)}{\pi}\right]^2 \mathrm{d}\tau}{\displaystyle\int_{-\infty}^{\infty} [1-|\tau|/T]^2\, \mathrm{d}\tau} = \frac{3}{2\pi^2}. \tag{83}$$

By Fourier transforming (79), we obtain

$$S_X^\alpha(f) = \begin{cases} \dfrac{1}{T^2} P(f+\alpha/2)P^*(f-\alpha/2) \\ \qquad \times \sum\limits_n S_A(f-\alpha/2-n/T), \\ \qquad \alpha = q/T, \\ 0, \quad \text{otherwise}, \end{cases} \tag{84}$$

where

$$P(f) = \int_{-\infty}^{\infty} p(t)\, \mathrm{e}^{-\mathrm{i}2\pi f t}\, \mathrm{d}t. \tag{85}$$

Thus, only spectral components that are separated by integer multiples of $1/T$ are correlated. This can be easily understood by expressing the process (76) as

$$X(t) = W(t) \otimes \frac{1}{T} p(t),$$

where

$$W(t) = A(t) \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi m t/T}.$$

Thus, each spectral component in $A(t)$ is shifted by all integer multiples m of $1/T$. As long as $S_A(f) = 0$ for $|f| > 1/2T$) and a flat pulse $p(t)$ [$p(t) = 0$ for $|t| > T/2$] is shown in Fig. 4.
components are perfectly correlated. That is, $|\rho_f^\alpha| = 1$ and, therefore, $\mathrm{DCS}_f^\alpha = 1$. A graph of the spectral correlation surface $|S_X^\alpha(f)|$ for a flat spectrum $S_A(f)$ ($S_A(f) = \text{constant}$ for $|f| \leqslant 1/2T$ and $S_A(f) = 0$ for $|f| > 1/2T$) and a flat pulse $p(t)$ [$p(t) = 0$ for $|t| > T/2$] is shown in Fig. 4.

The functions $R_X^\alpha(\tau)$ and $S_X^\alpha(f)$ are calculated in [5, 7] for other examples including sine waves that are phase- or frequency-modulated by stationary processes as well as by amplitude-modulated periodic pulse trains, and also periodic pulse trains that are pulse-width- or pulse-position-modulated.

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