On the Spectral Coherence of Nonstationary Processes

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Abstract—In this paper, it is shown that for a single sample path of a nonstationary process, reliable (i.e., low-variance) measurements of the degree of coherence between two spectral bands with substantial separation must be much smaller than 1/2 unless the nonstationarity is of a special type. That is, the nonstationarity must be either of a known form (prior to measurement) or of a periodic nature, which is known as cyclostationarity. This brings into question the utility of spectral coherence measurements on single sample paths of generally nonstationary processes.

I. Introduction

It is well known that a wide-sense stationary process does not exhibit spectral coherence. That is, the spectral representation of the process (its integrated Fourier transform) has uncorrelated increments. Or, in other words, if such a process is passed through two bandpass filters with nonoverlapping passbands, and the two filtered processes are frequency shifted to a common band, they will be uncorrelated. Furthermore, if a process has no spectral coherence, then it must be wide-sense stationary (cf. [1, sec. 3.6], [2, secs. 4.11 and 11.2]). This suggests that spectral coherence might be of some use in detecting and characterizing nonstationary processes and this has been proposed by some investigators (cf. [3]–[5]). However, if such characterizations are to be useful in practice, then the spectral coherence properties of nonstationary processes must be reliably measurable. This presents no particular problem in the rare application where an ensemble of sample paths of a nonstationary process is available for making ensemble-average measurements. However, in the much more common situation where only a single sample path of a nonstationary process is available and time-average measurements must be used, there are severe restrictions on the utility of spectral coherence measurements on nonstationary processes.

In particular, to be useful, a measurement of coherence must be reliable. That is, it must exhibit a sufficiently low level of variability from one sample path to another (i.e., small variance over the ensemble or, equivalently, a high probability of being approximately the same for each and every ensemble member or sample path). But it is shown in this paper that if such a measurement of spectral coherence is reliable, then the degree of coherence must be much smaller than 1/2 unless the nonstationarity is of a special type. That is, the reliably measured degree of spectral coherence must be very small unless the nonstationarity is either of a known form (prior to measurement) or is of a periodic nature, which is known as cyclostationarity.

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As a result of this restriction, spectral coherence measurements cannot provide useful statistics for detection and characterization of general nonstationarity.

II. Spectral Correlation

We consider a second-order nonstationary process $X(t)$ with autocorrelation function

$$R_X(t, \tau) = E\{X(t + \tau/2)X^*(t - \tau/2)\} \tag{1}$$

where $E\{\cdot\}$ denotes expectation and $(\cdot)^*$ denotes complex conjugation. We assume that $X(t)$ is sufficiently well behaved that the crosscorrelation function for the two bandpass-filtered and then frequency-shifted processes[1]

$$\begin{aligned} Y(t) &= \left[\int h_+(u)X(t-u)\,du\right]e^{-i2\pi(f+\alpha/2)t} \\ Z(t) &= \left[\int h_-(v)X(t-v)\,dv\right]e^{-i2\pi(f-\alpha/2)t} \end{aligned} \tag{2}$$

can be obtained as follows:

$$\begin{aligned} R_{YZ}(t, 0) &= E\{Y(t)Z^*(t)\} \\ &= E\left\{\int\int h_+(u)h_-^*(v)X(t-u)X^*(t-v) \right. \\ &\quad \left. \cdot e^{-i2\pi(f+\alpha/2)t}e^{+i2\pi(f-\alpha/2)t}\,du\,dv\right\} \qquad (3) \\ &= \int\int h_+(u)h_-^*(v)R_X\left(t - \frac{u+v}{2}, v-u\right) \\ &\quad \cdot du\,dv\,e^{-i2\pi\alpha t} \qquad (4) \end{aligned}$$

and similarly for $R_{YY}(t, 0)$ and $R_{ZZ}(t, 0)$. The filters of interest here are of the bandpass type and therefore have transfer functions

$$H_\pm(\nu) = \int h_\pm(t)e^{-i2\pi\nu t}\,dt \tag{5}$$

satisfying

$$H_\pm(\nu) = 0, \qquad |\nu - (f \pm \alpha/2)| > \Delta/2 \tag{6a}$$

for some bandwidth Δ and center frequencies $f \pm \alpha/2$. For example, for ideal bandpass filters, we have

$$H_\pm(\nu) = 1, \qquad |\nu - (f \pm \alpha/2)| < \Delta/2. \tag{6b}$$

[1]Unspecified limits of integration are understood to include the entire support of the integrand, which can be taken to be the entire real line $(-\infty, \infty)$.

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Consequently, $R_{YZ}(t, 0)$ is a measure of the correlation between the components of $X(t)$ whose spectral content (before being shifted to a common spectral band) resides in the two bands of width Δ centered at the two frequencies $f \pm \alpha/2$. We shall call $R_{YZ}(t, 0)$ the *spectral correlation*.

When only a single sample path of the process $X(t)$ is available, the spectral correlation $R_{YZ}(t, 0)$ can only be estimated and this can be done by replacing the expectation operation in (3) with a finite time-average operation

$$\hat{R}_{YZ}(t, 0) = \frac{1}{T}\int_{-T/2}^{T/2} Y(t+u)Z^*(t+u)\,du. \tag{7}$$

This estimate of spectral correlation can be related to estimates of instantaneous autocorrelations and instantaneous spectral densities for nonstationary processes, which have been considered in the literature (cf. [6] and [7, ch. 8] and references therein). Specifically, it is shown in [7, ch. 13] that the spectral correlation estimate (7) is approximately equal to the cyclic spectrum measurement

$$\hat{S}_X^\alpha(f) \triangleq \int_{-1/2\Delta}^{1/2\Delta} \hat{R}_X^\alpha(\tau)e^{-i2\pi f\tau}\,d\tau \tag{8}$$

where $\hat{R}_X^\alpha(\tau)$ is the cyclic autocorrelation measurement

$$\hat{R}_X^\alpha(\tau) \triangleq \frac{1}{T}\int_{-T/2}^{T/2} X(t+\tau/2)X^*(t-\tau/2)e^{-i2\pi\alpha t}\,dt \tag{9}$$

assuming that $T\Delta >> 1$ (cf. [9, ch. 12] and [7, ch. 11]). Furthermore, when $\alpha > \Delta$ and $Z << T$, then the measurement (8) is approximately equal to the Fourier coefficient

$$\hat{S}_X^\alpha(f) \cong \frac{1}{T}\int_{-T/2}^{T/2} \hat{S}_X(t, f)e^{-i2\pi\alpha t}\,dt \tag{10}$$

of the estimated instantaneous spectrum (cf. [7, ch. 8] and [9, ch. 12])

$$\hat{S}_X(t, f) \triangleq \int_{-1/2\Delta}^{1/2\Delta} \hat{R}_X(t, \tau)e^{-i2\pi f\tau}\,d\tau \tag{11}$$

(with spectral resolution Δ), where $\hat{R}_X(t, \tau)$ is the estimated instantaneous autocorrelation

$$\hat{R}_X(t, \tau) \triangleq \frac{1}{Z}\int_{-Z/2}^{Z/2} X(t+u+\tau/2)X^*(t+u-\tau/2)\,du. \tag{12}$$

Therefore, the problem of reliable measurement of spectral correlation is intimately related to the problem of reliable measurement of the instantaneous autocorrelation and instantaneous spectrum.

III. Cyclostationarity

We begin by observing that the average of the spectral correlation $R_{YZ}(t, 0)$ over all time t will be nonzero only if the quantity

$$\begin{aligned} R_X^\alpha(\tau) &= \langle R_X(t, \tau)e^{-i2\pi\alpha t}\rangle \\ &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t, \tau)e^{-i2\pi\alpha t}\,dt \end{aligned} \tag{13}$$

is not identically zero. This follows from the fact that

$$\langle R_{YZ}(t, 0)\rangle = \int\int h_+(u)h_-^*(v)\left\langle R_X\left(t - \frac{u+v}{2}, v-u\right) \cdot e^{-i2\pi\alpha t}\right\rangle du\,dv \tag{14}$$

where it has been assumed that $R_X(t, \tau)$ is well enough behaved to allow interchange of the order of the averaging operation and the two integrals.

The process $X(t)$ is said to *exhibit cyclostationarity* with *cycle frequency* α when the quantity (13) is not identically zero for some $\alpha \neq 0$ (cf. [9, sec. 12.1]). Thus, we see that no well-behaved process can exhibit a nonzero time-averaged value of spectral correlation unless it exhibits cyclostationarity. Furthermore, by interchanging the expectation and time-average operations, we see that the expected estimate of spectral correlation converges to zero

$$\lim_{T\to\infty} E\{\hat{R}_{YZ}(t, 0)\} = \langle R_{YZ}(t, 0)\rangle = 0 \tag{15}$$

if $X(t)$ does not exhibit cyclostationarity with cycle frequency α. Thus, there is an intimate relationship between spectral correlation measurements and cyclostationarity.

In order to obtain a reliable estimate of the spectral correlation (3) and (4) for all pairs of spectral bands, using only a single sample path of the process as in (7), it is required that we be able to obtain a reliable estimate of the autocorrelation function (1) using only a single sample path as in (12). It has been proven in [6] that this is possible only if

1) the process is cyclostationary or almost cyclostationary (at least locally) in the sense that the dependence of $R_X(t, \tau)$ on t is periodic or almost periodic (sums of periodic functions with incommensurate periods) for all τ, or
2) the process is locally stationary in the sense that $R_X(t, \tau)$ fluctuates very slowly in t for all τ (there exists a T such that for all τ $R_X(t, \tau)$ is nearly invariant in t throughout all intervals of length less than T, and the width of $R_X(t, \tau)$ in τ is much smaller than T for all t), or
3) the process has a form of nonstationarity that is known *a priori* ($R_X(t, \tau)$ is composed of functions of t alone, all of which are known *a priori* and functions of τ alone); e.g.,

$$R_X(t, \tau) = \sum_{k=1}^{n} \phi_k(t)\theta_k(\tau) \tag{16}$$

for some known functions $\phi_k(t)$ (cf. [6]).

The connection between cyclostationarity and spectral correlation is a strong one and is studied in great detail in [7]–[9], where numerous examples of specific models that exhibit cyclostationarity are studied. Also, the required ergodic properties for case 1) are developed in [9, sec. 12.7] and [10]. When $X(t)$ exhibits the appropriate ergodic property (called a *cycloergodic* property), then all that is typically required for $\hat{R}_{YZ}(t, 0)$ to be a reliable estimate of $R_{YZ}(t, 0)$ is that the product of averaging time T and spectral resolution bandwidth Δ be sufficiently large:

$$T\Delta >> \frac{\sqrt{R_{YY}(t, 0)R_{ZZ}(t, 0)}}{|R_{YZ}(t, 0)|}. \tag{17}$$

In obtaining this particular condition, it has been assumed that the bandwidth parameter Δ from (6) is small enough to accurately resolve the spectral coherence properties of the process [7, ch. 15], [11].

The situation in which the form of nonstationarity is not periodic (or almost periodic) but is known prior to measurement is a relatively rare occurrence in practice. Moreover, when the form of nonstationarity is known in advance, there is no longer a need to detect or characterize the nonstationarity. Therefore, we focus our attention on the one remaining case of interest: locally stationary processes.

IV. Local Stationarity

A process can be locally stationary [12] only if an appropriate measure of the bandwidth B_0 of the nonstationarity (fluctuations of $R_X(t, \tau)$ in t for all τ) is much smaller than the bandwidth B_* of the fluctuations of $R_X(t, \tau)$ in τ for all t; that is, B_0 must be much smaller than the width in f of the instantaneous spectrum

$$S_X(t, f) = \int R_X(t, \tau) e^{-i2\pi f\tau} \, d\tau. \tag{18}$$

The extreme limit of a locally stationary process corresponds to an idealized autocorrelation that is a time-dependent Dirac delta $\delta(t, \tau)$ corresponding to nonstationary white noise.

There are primarily two ways that nonstationary white noise can be generated. One is by applying a time-variant time-scale transformation $b(t)$ to stationary white noise $N(t)$, to obtain $X(t) = N[b(t)]$, and the other is by applying a time-variant amplitude-scale transformation $a(t)$ to stationary white noise $N(t)$ to obtain $X(t) = a(t)N(t)$. The autocorrelation that results from the first type of nonstationary white noise is

$$\begin{aligned} R_X(t, \tau) &= R_N[b(t + \tau/2) - b(t - \tau/2)] \\ &= \delta[b(t + \tau/2) - b(t - \tau/2)]. \end{aligned} \tag{19}$$

Since $b(t)$ must be an invertible function, then the only value of τ for which the argument of $\delta[\cdot]$ is zero is $\tau = 0$; and in the limit at $\tau \to 0$, we obtain

$$\lim_{\tau \to 0} \frac{1}{\tau} [b(t + \tau/2) - b(t - \tau/2)] = \frac{db(t)}{dt}.$$

Therefore, we have

$$R_X(t, \tau) = \delta\left[\frac{db(t)}{dt} \tau\right] = \frac{1}{|db(t)/dt|} \delta(\tau) \tag{20}$$

where the second equality results from a basic property of the Dirac delta.

The autocorrelation for the other type of white noise is

$$\begin{aligned} R_X(t, \tau) &= a(t + \tau/2) a(t - \tau/2) R_N(\tau) \\ &= a(t + \tau/2) a(t - \tau/2) \delta(\tau). \end{aligned} \tag{21}$$

Since $\delta(\tau) \neq 0$ only for $\tau = 0$, then we obtain

$$R_X(t, \tau) = a^2(t) \delta(\tau). \tag{22}$$

Consequently, both types of nonstationary white noise have the same form of autocorrelation (22) (by identifying $a^2(t)$ with $|db(t)/dt|^{-1}$).

Since $B_* = \infty$ for the idealized model (22), then $B_0 << B_*$, as desired, regardless of how large B_0 is (as long as it is finite). For the model (22), B_0 is simply the bandwidth of $a^2(t)$. Although $a(t)$ in the model $X(t) = a(t)N(t)$ is considered to be nonstochastic, in the sense that every ensemble member of $X(t)$ contains the same factor $a(t)$, there is no reason that we cannot let $a(t)$ be a sample path of some stochastic process. This is, in fact, done in the next section.

A more practical model for a locally stationary process can be obtained by replacing $\delta(\tau)$ in (22) with a well behaved autocorrelation function $R_N(\tau)$ that is very narrow in width. That is, the bandwidth B_* of the corresponding spectrum

$$S_N(f) = \int R_N(\tau) e^{-i2\pi f\tau} \, d\tau \tag{23}$$

must be much broader than the bandwidth B_0 of $a^2(t)$. A physical situation that could give rise to this type of model is a thermal noise source with a time-varying resistance. Since the spectral intensity is proportional to the resistance, but the bandwidth is independent of the resistance, then, with the time-variant resistance proportional to $a^2(t)$, we have the model $R_X(t, \tau) = a^2(t) R_N(\tau)$.

V. Evaluation of Spectral Coherence

Using the idealized model (22) for a locally stationary process, we see that the expression (4) for the spectral correlation reduces to the downconverted convolution

$$R_{YZ}(t, 0) = \int g(u) a^2(t - u) \, du \, e^{-i2\pi\alpha t} \tag{24}$$

where

$$g(u) = h_+(u) h_-^*(u). \tag{25}$$

The transfer function $G(\nu)$ corresponding to this convolution is (from (6a))

$$G(\nu) = \int H_+(\nu + \mu) H_-^*(\mu) \, d\mu = 0, \qquad |\nu - \alpha| > \Delta. \tag{26}$$

Thus, only spectral components of $a^2(t)$ in the band of width 2Δ centered at α contribute to the spectral correlation $R_{YZ}(t, 0)$.

If $a^2(t)$ contains a finite-strength additive sine-wave component with frequency α, then the spectral correlation (24) can be substantial. However, this corresponds to the case of a process that exhibits cyclostationarity with cycle frequency α. It appears that the only type of locally stationary processes that does not exhibit cyclostationarity but that does exhibit substantial spectral correlation is that which is locally cyclostationary in the sense that $a^2(t)$ is highly oscillatory with a center frequency near α.

To quantify this observation, we can study how close the spectral coherence (spectral correlation coefficient)

$$\rho = \frac{|R_{YZ}(t, 0)|}{\sqrt{R_{YY}(t, 0) R_{ZZ}(t, 0)}} \tag{27}$$

can be to its maximum possible value of unity. It follows from (22) and the counterparts of (4) for $R_{YY}(t, 0)$ and $R_{ZZ}(t, 0)$ that

$$\begin{aligned} R_{YY}(t, 0) &= \int g_+(u) a^2(t - u) \, du \\ R_{ZZ}(t, 0) &= \int g_-(v) a^2(t - v) \, dv \end{aligned} \tag{28}$$

where the corresponding transfer functions are (from (6a))

$$G_{\pm}(\nu) = \int H_{\pm}(\nu + \mu) H_{\pm}^{*}(\mu)\, d\mu = 0, \qquad |\nu| > \Delta. \tag{29}$$

However, we have the difficulty that this spectral coherence (27) varies with time t in a way that depends on the particular nonstationarity.

One approach to obtaining a unique numerical measure of spectral coherence is to use the time-average root-mean-square values of the autocorrelation and crosscorrelation, e.g.,

$$\langle |R_{YZ}(t, 0)|^2 \rangle^{1/2} = \left[\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |R_{YZ}(t, 0)|^2\, dt \right]^{1/2}$$

to obtain

$$\bar{\rho} = \frac{\langle |R_{YZ}(t, 0)|^2 \rangle^{1/2}}{\langle |R_{YY}(t, 0)|^2 \rangle^{1/4} \langle |R_{ZZ}(t, 0)|^2 \rangle^{1/4}}. \tag{30}$$

This requires that the nonstationarity be persistent (that the process neither blow up nor die out as $t \to \infty$; i.e., as $t \to \infty$, neither $R_X(t, \tau) \to 0$ nor $R_X(t, \tau) \to \infty$ is allowed). It follows from (24) that

$$\langle |R_{YZ}(t, 0)|^2 \rangle = \iint g(u) g^{*}(v) \langle a^2(t-u) a^2(t-v) \rangle\, du\, dv \tag{31}$$

assuming that $a^2(t)$ is well behaved so that the order of the integrals and the time-average operation can be interchanged. Also, it follows from (28) that

$$\langle |R_{YY}(t, 0)|^2 \rangle = \iint g_{+}(u) g_{+}^{*}(v) \langle a^2(t-u) a^2(t-v) \rangle\, du\, dv \tag{32}$$

and similarly for $\langle |R_{ZZ}(t, 0)|^2 \rangle$.

Let us consider the case where $a(t)$ has a multivariate Gaussian fraction-of-time distribution [7] (e.g., $a(t)$ is a sample path of an ergodic stationary Gaussian process [9]) with time-average spectrum $S_a(\nu)$. Then (31) reduces to (see [9, ch. 10, exc. 37])

$$\langle |R_{YZ}(t, 0)|^2 \rangle = \int \left\{ \left[\int S_a(\nu)\, d\nu \right]^2 \delta(\mu) + 2 \int S_a(\mu - \nu) S_a(\nu)\, d\nu \right\} |G(\mu)|^2\, d\mu. \tag{33}$$

The same formula applies to $\langle |R_{YY}(t, 0)|^2 \rangle$ and $\langle |R_{ZZ}(t, 0)|^2 \rangle$ except that G must be replaced with G_{+} and G_{-}, respectively.

We consider two cases, one where $a(t)$ is highly oscillatory (locally cyclostationary $X(t)$) and one where $a(t)$ has equal spectral content throughout some band centered at zero frequency, and is therefore nonoscillatory

Case 1: Oscillatory Nonstationarity: Substituting the spectrum

$$S_a(\nu) = \begin{cases} S_0, & \left| |\nu| - \alpha/2 \right| < \epsilon/2 \\ 0, & \text{otherwise} \end{cases} \tag{34}$$

into (33) and its counterpart for (32), assuming that $\Delta \geq \epsilon$, and using (6), (26), (29), and (30), yields the result

$$\bar{\rho} = \frac{\left[\frac{1}{2} - \frac{1}{3} \frac{\epsilon}{\Delta} + \frac{1}{12} \left(\frac{\epsilon}{\Delta} \right)^2 \right]^{1/2}}{\left[2 - \frac{2}{3} \frac{\epsilon}{\Delta} + \frac{1}{6} \left(\frac{\epsilon}{\Delta} \right)^2 \right]^{1/2}} \to \frac{1}{2} \qquad \text{as } \epsilon \to 0 \tag{35}$$

independent of f and α. Thus, we have a relatively high degree of spectral coherence in this case, regardless of the amount of spectral separation $\alpha - \Delta$ (i.e., the separation between the two bands whose correlation is measured is $\alpha - \Delta$). For comparison, (27) yields $\rho = 1/2$ for $a(t) = \cos(\pi \alpha t)$ in which case $X(t)$ is cyclostationary. Also, if $S_N(f) = 0$ for $|f| > \alpha/2$, and $a(t) = \cos(\pi \alpha t)$, then $\rho = 1$ for all $\Delta < \alpha/2$ [7], [9].

Case 2: Nonoscillatory Nonstationarity: Substituting the spectrum

$$S_a(\nu) = \begin{cases} S_0, & |\nu| < B_0 \\ 0, & \text{otherwise} \end{cases} \tag{36}$$

into (33) and its counterpart for (32), and using (6), (26), (29), and (30), yields

$$\bar{\rho} = \left[\frac{\frac{\Delta}{B_0} \left(1 - \frac{\alpha}{2B_0} \right)}{3 + \frac{2\Delta}{B_0} - \left(\frac{\Delta}{2B_0} \right)^2} \right]^{1/2}, \qquad \Delta < B_0 \tag{37}$$

for $0 \leq \alpha \pm \Delta \leq 2B_0$, independent of f. In order to obtain a pair of spectral bands with substantial separation, we require $\alpha \gg \Delta$ and, consequently, $\Delta \ll B_0$. In this case, we obtain

$$\bar{\rho} \cong \left[\frac{\Delta}{B_0} \left(1 - \frac{\alpha}{2B_0} \right) \right]^{1/2} \ll 1. \tag{38}$$

Thus, the degree of spectral coherence is very small. The only way to obtain a substantial value of $\bar{\rho}$ is to completely forfeit spectral separation by choosing $\Delta = B_0$ and $\alpha = B_0$. Then $R_{YZ}(t, 0)$ is the correlation of the entire positive-frequency spectral band ($0 \leq f \leq B_0$) with the corresponding negative-frequency band. This yields the largest possible value of $\bar{\rho}$ for this case 2

$$\bar{\rho} = \sqrt{\frac{2}{19}} \cong 0.33.$$

As another example, if the two spectral bands of width Δ are separated by a minimal nonzero amount, say Δ, then $\alpha = 2\Delta$ and we require $\Delta \leq 2B_0/3$ in order for these spectral bands to remain within the support band $[-B_0, B_0]$. In this case, (37) reduces to

$$\bar{\rho} = \sqrt{\frac{1}{19}} \cong 0.23.$$

Thus, any substantial spectral separation will result in a relatively small degree of coherence (much smaller than $1/4$).

VI. Simulations

To corroborate the theoretical predictions in the previous sections, computer simulations were conducted to evaluate an empirical spectral correlation coefficient corresponding to the probabilistic spectral correlation coefficient (27). In order to obtain a computationally efficient simulation, the empirical spectral correlation (7) was implemented using an FFT algorithm. That is, the filtering and downconversion (2) was implemented using an FFT and then the fact that time-averaging products of short FFT's, as in (7), is approximately equivalent to frequency smoothing the product of long FFT's was used [7]. Thus, the spectral correlation measurement approximating (7) (for fixed t) that was implemented is given by

$$\hat{R}_{YZ}(t, 0) \cong \hat{S}_X^\alpha(f)_M \triangleq \frac{1}{M} \sum_{m=-M/2}^{M/2-1} \tilde{X}(f + mF_s + \alpha/2) \cdot \tilde{X}^*(f + mF_s - \alpha/2) \tag{39}$$

where M (an even integer) is the number of frequency bins averaged in the frequency-smoothing operation, $F_s = 1/NT_s$ is the discrete frequency increment, and $\tilde{X}(f)$ is the FFT

$$\tilde{X}(f) = \sum_{k=0}^{N-1} X(kT_s) e^{-i2\pi fkT_s} \tag{40}$$

where T_s is the time-sampling increment. Similarly,

$$\hat{R}_{YY}(t, 0) \cong \hat{S}_X^0(f + \alpha/2)_M. \tag{41}$$

The spectral correlation coefficient computed is given by

$$\hat{\rho}_X^\alpha(f)_M = \frac{\left|\hat{S}_X^\alpha(f)_M\right|}{\sqrt{\hat{S}_X^0(f + \alpha/2)_M \hat{S}_X^0(f - \alpha/2)_M}} \tag{42}$$

and is the empirical counterpart of the probabilistic correlation coefficient (27).

The spectral resolution bandwidth of this measurement is $\Delta = MF_s$, and the product of averaging time $T = NT_s$ and spectral resolution bandwidth Δ is given by

$$T\Delta = (NT_s)(MF_s) = M. \tag{43}$$

For a reliable measurement, it is required that

$$M \gg 1/\rho > 1 \tag{44}$$

which follows from (17) and (27). From this point forward, only $T_s = 1$ is considered.

Attention is focused on processes of the form

$$X(t) = a(t)N(t) \tag{45}$$

where $N(t)$ is white Gaussian noise with bandwidth B_* (possibly low-pass filtered) and $a(t)$ is a deterministic function (possibly a fixed sample path of some stochastic process). Specifically, the following cases are considered:

case 1: $a(t)$ is a constant ($X(t)$ is stationary);

case 2: $a(t)$ is a sample path of nonfiltered white Gaussian noise ($X(t)$ is nonstationary and nonlocally stationary);

case 3: $a(t)$ is a sample path of low-pass filtered white Gaussian noise with bandwidth $B_0 = 1/16 = (1/8)B_*$ ($X(t)$ is locally stationary and the nonstationarity is nonoscillatory);

case 4: $a(t)$ is a sample path of bandpass filtered white Gaussian noise with bandwidth $\epsilon = 1/1024$ and center frequency $f_0 = 1/16$ ($X(t)$ is locally stationary and the nonstationarity is oscillatory; i.e., $X(t)$ is locally cyclostationary);

case 5: $a(t)$ is a sine wave with frequency $f_0 = 1/16$ ($X(t)$ is cyclostationary and $\rho = 1/2$);

case 6: $a(t)$ is a sine wave with frequency $f_0 = 1/16$, and $N(t)$ is band limited to a bandwidth of $B^* = 1/32$ ($X(t)$ is cyclostationary and $\rho = 1$).

In all but case 6, $N(t)$ is unfiltered white noise ($B_* = 1/2$). For cases 5 and 6 (with $a(t) = \cos(2\pi f_0 t + \phi_0)$), the idealized measurement (8), with $T \to \infty$ and then $\Delta \to 0$, is equal to [7, ch. 12], [9, ch. 12]

$$\lim_{\Delta\to 0} \lim_{T\to\infty} \hat{S}_X^\alpha(f) = \begin{cases} \frac{1}{4}S_N(f - f_0) + \frac{1}{4}S_N(f + f_0), & \alpha = 0 \\ \frac{1}{4}S_N(f)e^{\pm i2\phi_0}, & \alpha = \pm 2f_0 \\ 0, & \alpha \neq 0, \pm 2f_0. \end{cases} \tag{46}$$

Therefore, the only appropriate value of $\alpha \neq 0$ is $\alpha = 2f_0$ (or $\alpha = -2f_0$), and for low-pass $N(t)$ where $S_N(f)$ peaks at $f = 0$ (case 6), the most appropriate value of f is $f = 0$, whereas for white $N(t)$ (case 5) the value of f is irrelevant.

For each of the six cases, the spectral correlation coefficient $\hat{\rho}_X^\alpha(f)_M$ is computed for certain values of α, f, and M. For each set of values of these parameters, $\rho_X^\alpha(f)_M$ is computed for 100 sample paths of $N(t)$ (and one fixed choice of $a(t)$), and the ensemble mean and variance of $\hat{\rho}_X^\alpha(f)_M$ are computed. For cases 4–6, the only appropriate value of α is $\alpha = 2f_0 = 1/8$, and the value $f = 0$ is appropriate in all three of these cases. For cases 1 and 2, the value of α is irrelevant, so $\alpha = 1/8$ and $f = 0$ are again used. However, for case 3, the values of α and f could possibly be relevant, so the representative pairs $(\alpha, f) = (1/64, 0)$, $(1/64, 1/64)$, $(1/32, 0)$, $(1/32, 1/32)$, $(1/16, 0)$ are considered. The value of Δ is not significant since $N(t)$ is white in all but case 6. Nevertheless, to insure that the two spectral bands whose correlation is measured are always nonoverlapping, the value of $\Delta = 1/64$ is chosen for all cases.

The results of the simulations are recorded in Table I. It is clear from these results that in cases 1–3, where there is no cyclostationarity or local cyclostationarity, $\hat{\rho}_X^\alpha(f)_M$ always converges to insignificant values ($\ll 1$) as M is increased in an attempt to increase reliability. For case 4, where there is local cyclostationarity, the same is true, but the convergence is slower. For example, for $M = 16$, the mean value of $\hat{\rho}_X^\alpha(f)$ is significant ($\cong 1/2$) and there is some reliability (variance $<$ $(1/2)$ mean^2). For cases 5 and 6, where there *is* cyclostationarity, $\hat{\rho}_X^\alpha(f)$ converges to significant values with substantial reliability.

VII. Summary and Conclusion

After defining spectral correlation and relating it to the instantaneous spectrum of a nonstationary process in Section II, it is shown in Section III that the time-averaged value of spectral correlation is zero for all nonstationary processes except those that exhibit cyclostationarity. This is equivalent to saying that the expected value of the empirical spectral correlation converges to zero as the averaging time used to measure the correlation grows without bound, unless the process exhibits cyclostationarity. It is also concluded from mathematical demonstrations reported in [6] that the only situations in which spectral correlation can be reliably measured from a single sample path are those in which the process exhibits cyclostationarity (at least locally), or is locally stationary, or the form of nonstationarity is known in advance of measurement, a case of no relevance for techniques designed to detect and characterize

TABLE I
MEANS AND VARIANCES OF $\hat{\rho}_X^\alpha(f)$ FOR ENSEMBLE SIZE OF 100 (SPECTRAL RESOLUTION BANDWIDTH IS $\Delta = 1/64$ AND AVERAGING TIME IS $N = M/\Delta = 64M$)

Case	(α, f)	M = 4	M = 16	M = 64	M = 256	M = 1024
Case 1	(1/8, 0)					
mean		0.599	0.290	0.155	0.065	0.025
variance		0.234	0.143	0.069	0.032	0.014
Case 2	(1/8, 0)					
mean		0.531	0.286	0.155	0.080	0.044
variance		0.268	0.139	0.075	0.042	0.022
Case 3a	(1/64, 0)					
mean		0.597	0.321	0.177	0.086	0.044
variance		0.232	0.149	0.103	0.046	0.024
Case 3b	(1/64, 1/64)					
mean		0.412	0.269	0.130	0.068	0.033
variance		0.177	0.123	0.067	0.036	0.017
Case 3c	(1/32, 0)					
mean		0.636	0.301	0.170	0.087	0.053
variance		0.244	0.163	0.090	0.043	0.027
Case 3d	(1/32, 1/32)					
mean		0.423	0.252	0.125	0.063	0.037
variance		0.179	0.138	0.066	0.346	0.018
Case 3e	(1/16, 0)					
mean		0.589	0.321	0.165	0.082	0.043
variance		0.261	0.162	0.077	0.043	0.024
Case 4	(1/8, 0)					
mean		0.677	0.482	0.264	0.113	0.060
variance		0.214	0.186	0.122	0.061	0.029
Case 5	(1/8, 0)					
mean		0.645	0.520	0.490	0.493	0.494
variance		0.249	0.159	0.107	0.058	0.027
Case 6	(1/8, 0)					
mean		0.788	0.932	0.984	0.997	0.999
variance		0.130	0.053	0.016	0.003	0.001

nonstationarity. Knowing that reliable spectral correlation measurement is indeed possible and useful for processes that exhibit cyclostationarity [7], this leaves only the case of locally stationary processes for further investigation. After characterizing the class of locally stationary processes in Section IV, it is analytically verified in Section V that the spectral correlation coefficient must be much smaller than unity unless the locally stationary process also exhibits local cyclostationarity. Finally, in Section VI, these theoretical predictions are corroborated with simulations. It is shown that for stationary processes, locally stationary processes, and nonstationary processes that are neither locally stationary nor locally cyclostationary, measurements of empirical spectral correlation are insignificant: the spectral correlation coefficient is either much smaller than unity or highly unreliable (or both). And, it is shown that for processes that exhibit cyclostationarity or local cyclostationarity, reliable measurement of substantial spectral correlation is indeed possible.

It is concluded that spectral correlation measurements on single time series are of highly questionable value for detection and characterization of nonstationarity, except in the special cases of cyclostationarity. More generally, it can be concluded that spectral coherence properties of generally nonstationary processes, those that do not exhibit cyclostationarity or local cyclostationarity, are properties of ensembles only; they are not properties of single time series. Consequently, they cannot in any way be reliably exploited for statistical signal processing tasks involving only single-sample-path processing.

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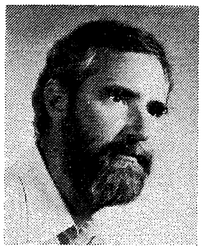

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