Identification of Systems with Cyclostationary Input and Correlated Input/Output Measurement Noise

WILLIAM A. GARDNER

Abstract—**Conventional statistical methods of system identification, such as Wiener's modeling method, can perform poorly when input/output measurements are severely corrupted, especially if the corruption is correlated from input to output. A new approach that is applicable when the uncorrupted system-input is cyclostationary can, in principle, provide ideal system identification regardless of measurement corruption, provided that the corruption is not also cyclostationary with all the same cycle frequencies. This new method of corruption-tolerant system identification is introduced in this note. An application to interference-tolerant TDOA estimation is described and illustrated with simulations.**

I. Introduction

A common approach to identifying the transfer function for a linear time-invariant model of some physical input–output system is based on Wiener's optimum filtering theory for stationary random time series. However, this approach can perform poorly when the measurements of the system output and, especially, input are corrupted with noises or interferences, particularly if the input and output corruption are correlated with each other. The purpose of this note is to present a new alternative to Wiener's approach that is applicable when the system input exhibits cyclostationarity. This alternative approach can, in principle, eliminate performance degradation due to input/output corruption.

The problem of correlated input/output corruption typically arises when the input and output of the physical system to be identified are not

Manuscript received May 25, 1988; revised December 15, 1988. Paper recommended by Past Associate Editor, M. Gevers. This work was supported by ESL, Inc., with partial matching support from the California State MICRO Program.

The author is with the Department of Electrical Engineering and Computer Science, University of California, Davis, CA 95616.

IEEE Log Number 8933394.

0018-9286/90/0400-0449$01.00

directly accessible or are not fully under the control of the experimenter. For instance, the physical system might be a remote wave-propagating medium which is subject to excitation from waves other than that which the experimenter transmits or is otherwise able to measure. Also, if the sensors for measuring the input and output of the system are exposed to excitation from some common interfering source, then the input–output corruption will be correlated. This is illustrated at the end of this note with an application of system identification methods to the problem of interference-tolerant time-difference-of-arrival estimation.

The system identification method introduced in this note can be explained within the framework of stationary and cyclostationary stochastic processes (cf. [1]–[3]) or within the nonstochastic framework of persistent time series that exhibit stationarity and cyclostationarity properties (cf. [4] and [5]). For example, the original theory of optimum extrapolation, interpolation, and smoothing of time series introduced by Wiener [6] was developed within the nonstochastic framework of persistent time series that exhibit stationarity, whereas more recent accounts of this theory are presented within the framework of stationary stochastic processes (cf. [3]). In order to minimize unnecessary abstractions, the system identification method introduced here is explained in terms of nonstochastic time series. The primary assumption required is that the cross correlation[1]

$$\langle p(t)q^*(t)\rangle \triangleq \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} p(t)q^*(t)\,dt \tag{1}$$

exist for all time series $p(t)$ and $q(t)$ obtained from time shifts and/or frequency shifts of the unknown system input $z(t)$, input-measurement noise $n(t)$, output $w(t)$, and output-measurement noise $m(t)$. For example, for $p(t) = w(t)$ and a time shift of τ and a frequency-shift of α applied to the time series $z(t)$, we have $q(t) = z(t-\tau)\exp(+i2\pi\alpha t)$ and

$$\langle p(t)q^*(t)\rangle = \langle w(t)z(t-\tau)\exp(-i2\pi\alpha t)\rangle \triangleq R_{wz}(\tau;\alpha). \tag{2}$$

If $R_{wz}(\tau; 0) \not\equiv 0$, then $w(t)$ and $z(t)$ are said to exhibit joint stationarity. If $R_{wz}(\tau;\alpha)\not\equiv 0$ for some $\alpha \neq 0$, then $w(t)$ and $z(t)$ are said to exhibit joint cyclostationary [1]–[5] with cycle frequency α because the lag-product waveform $w(t)z(t-\tau)$ contains a finite additive sine-wave component with frequency α and with complex strength given by the Fourier coefficient (2).

A simple example of a time series that exhibits cyclostationarity is $z(t) = a(t)\cos(\omega t)$, where $a(t)$ is a sample path of a zero-mean ergodic stationary stochastic process. Although $z(t)$ does not contain finite additive sine-wave components, its lag product waveforms $z(t)z(t-\tau)$ do for each value of τ for which $\langle a(t)a(t-\tau)\rangle \neq 0$ (see Section III). (Other examples are given in [1]–[5].)

Inherent in the assumption that the correlations (1) involving the output $w(t)$ exist is the requirement that the system be stable. Moreover, it will be assumed that the system is linear and time invariant. However, the method can work well for small departures from linearity and slow time variations.

A basic assumption of the system identification method introduced in this note is that the system input exhibit cyclostationarity ($R_{zz}(\tau;\alpha)\not\equiv 0$) with a *known* cycle frequency α. In practice, α needs to be known only to within a tolerance $\Delta\alpha$ that does not exceed the reciprocal of the integration time T used to measure the correlations used by the method: $\Delta\alpha < 1/T$. Moreover, if the system input exhibits cyclostationarity, but the cycle frequency α is unknown, it can be measured by detecting the spectral line (at α) in the spectrum of a lag-product waveform obtained from the noise-corrupted input $x(t) = z(t) + n(t)$ [5], [7].

Another assumption is that the Fourier transform (cross spectrum),

$$S_{zz}(f;\alpha) \triangleq \int_{-\infty}^{\infty} R_{zz}(\tau;\alpha)\exp(-i2\pi f\tau)\,d\tau, \tag{3}$$

of the cross correlation $R_{zz}(\tau;\alpha)$ exist and be nonzero throughout the band of frequencies $f \in [f_1-\alpha, f_2-\alpha]$, where $[f_1, f_2]$ is the band over which the transfer function, $H_*(f)$, of the unknown system is to be identified. Examination of the extensive variety of examples of types of time series that exhibit cyclostationarity in [5] reveals that this assumption is not exceedingly restrictive. For example, a necessary and often sufficient condition is that the autospectrum of $z(t)$ be nonzero at the two frequencies f and $f-\alpha$ for all $f \in [f_1, f_2]$. A specific example is given in Section III.

The final assumption is that the input- and output-measurement noises have zero average value, are independent of the system input, and do not exhibit cyclostationarity with the same cycle frequency α as that of the input.

Since the system identification method introduced in this note is a variation on Wiener's classical method of dividing the measured input/output cross spectrum by the input autospectrum, we begin with a brief review of Wiener's method and its drawbacks.

II. Wiener's System Model

Let $h_*(t)$ and $H_*(f)$ be the unknown impulse-response function and transfer function for a linear and time-invariant system

$$H_*(f) = \int_{-\infty}^{\infty} h_*(t)\exp(-i2\pi ft)\,dt. \tag{4}$$

Let $z(t)$ be the input to this system and let $w(t)$ be the corresponding output. These are related by the convolution

$$w(t) = \int_{-\infty}^{\infty} h_*(t-u)z(u)\,du. \tag{5}$$

The actual measurements of this input and output are given by

$$x(t) = z(t) + n(t)$$

$$y(t) = w(t) + m(t), \tag{6}$$

where $n(t)$ and $m(t)$ represent additive measurement corruption. In terms of these measurements we have, from (5) and (6),

$$y(t) = \int_{-\infty}^{\infty} h_*(t-u)[x(u)-n(u)]\,du + m(t). \tag{7}$$

To understand Wiener's method for identifying the system impulse-response function or transfer function, we envision a system model with transfer function $H(f)$, and we consider using the measured corrupted input $x(t)$ as the input to this model to obtain the model output

$$\hat{y}(t) = \int_{-\infty}^{\infty} h(t-u)x(u)\,du. \tag{8}$$

We then solve for the particular model $H(f)$ that minimizes the time-averaged squared error between the actual system's measured output $y(t)$ and the model's output $\hat{y}(t)$. The solution is given by Wiener's model [6]

$$H(f) = \frac{S_{yx}(f)}{S_{xx}(f)}, \tag{9}$$

where $S_{yx}(f)$ is the cross spectrum of $y(t)$ and $x(t)$, and $S_{xx}(f)$ is the autospectrum of $x(t)$:

$$S_{yx}(f) \equiv S_{yx}(f;0) = \int_{-\infty}^{\infty} R_{yx}(\tau;0)\exp(-i2\pi f\tau)\,d\tau. \tag{10}$$

To apply this result (9), we would use the input/output measurements $x(t)$ and $y(t)$ to estimate the spectra in (9), and then form a ratio of these spectral estimates as indicated in (9).

It follows from (7) that

$$S_{yx}(f) = H_*(f)[S_x(f) - S_{nx}(f)] + S_{mx}(f), \tag{11}$$

[1] The theory presented in this note could just as well be based on one-sided time averages obtained from (2) by replacing the averaging interval $[-T/2, T/2]$ with $[0, T]$.

and it follows from (6) that

$$S_x(f) = S_z(f) + S_n(f)$$

$$S_{nx}(f) = S_n(f)$$

$$S_{mx}(f) = S_{mn}(f), \tag{12}$$

assuming that $z(t)$ and $n(t)$, and also $z(t)$ and $m(t)$, are orthogonal, i.e., $R_{zn}(\tau) \equiv R_{zm}(\tau) \equiv 0$. Thus, (9), (11), and (12) yield

$$H(f) = H_*(f)\left[1 - \frac{S_n(f)}{S_z(f)+S_n(f)}\right] + \frac{S_{mn}(f)}{S_z(f)+S_n(f)}. \tag{13}$$

We can see that the model $H(f)$ coincides with the actual system $H_*(f)$ only if the input corruption $n(t)$ is absent: $S_n(f) \equiv 0$. When it is present, the degradation in modeling performance is even greater if the input/output corruption is correlated: $S_{nm}(f) \not\equiv 0$.

III. The SPECCORR Method

Using the definition

$$R_{yx}(\tau;\alpha) \triangleq \langle y(t)x(t-\tau)\exp(-i2\pi\alpha t)\rangle$$

and the model (6)–(7), together with the assumptions stated in Section I, we obtain

$$R_{yx}(\tau;\alpha) = \int_{-\infty}^{\infty} h_*(u)\exp(-i2\pi\alpha u)[R_{xx}(\tau-u;\alpha) - R_{nx}(\tau-u;\alpha)]\,du + R_{mx}(\tau;\alpha), \tag{14}$$

where

$$R_{nx}(\tau;\alpha) = R_{nn}(\tau;\alpha) + R_{nz}(\tau;\alpha)$$

$$R_{mx}(\tau;\alpha) = R_{mn}(\tau;\alpha) + R_{mz}(\tau;\alpha). \tag{15}$$

Given that $z(t-\tau)\exp(+i2\pi\alpha t)$ is orthogonal to $n(t)$ and $m(t)$ $(R_{nz}(\tau;\alpha) \equiv R_{mz}(\tau;\alpha) \equiv 0)$ and that neither $n(t)$ nor $m(t)$ exhibits cyclostationarity with cycle frequency α $(R_{nn}(\tau;\alpha) \equiv R_{mn}(\tau;\alpha) \equiv 0)$, as assumed in Section I, we have $R_{nx}(\tau;\alpha) \equiv R_{mx}(\tau;\alpha) \equiv 0$. Consequently, (14) reduces to

$$R_{yz}(\tau;\alpha) = \int_{-\infty}^{\infty} h_*(u)\exp(-i2\pi\alpha u)R_{xx}(\tau-u;\alpha)\,du. \tag{16}$$

Fourier transforming both sides of (16) yields

$$S_{yx}(f;\alpha) = H_*(f+\alpha)S_{xx}(f;\alpha), \tag{17}$$

which can be solved for the unknown transfer function:

$$H_*(f) = \frac{S_{yx}(f-\alpha;\alpha)}{S_{xx}(f-\alpha;\alpha)}, \qquad f \in [f_1, f_2]. \tag{18}$$

Since $S_{nn}(f;\alpha) \equiv 0$, then $S_{xx}(f;\alpha) \equiv S_{zz}(f;\alpha)$ and the denominator in (18) is therefore nonzero (as assumed in Section I).

We see from (18) that, like Wiener's modeling method (9), we need only measure two spectra and form their ratio. However, unlike Wiener's method, we can, in principle, perfectly identify the unknown system provided only that the input exhibits appropriate cyclostationarity with some cycle frequency α and the input/output measurement noise does not exhibit cyclostationarity with the same cycle frequency α.

This method of system identification is called the SPECCORR (SPECtral CORrelation Ratio) method since the cross spectra in the ratio (18) are actually spectral correlation functions (cf. [3]–[5]). It is noted that with $\alpha = 0$, the SPECCORR method (18) reduces to Wiener's method (9).

Example: We consider the time series

$$z(t) = a(t)\cos(\omega t) = \frac{1}{2}a(t)\exp(i\omega t) + \frac{1}{2}a(t)\exp(-i\omega t), \tag{19}$$

where $a(t)$ exhibits stationarity but not cyclostationarity. Then we have

$$\begin{aligned}R_{zz}(\tau;\alpha) &= \langle z(t)z(t-\tau)\exp(-i2\pi\alpha t)\rangle\\ &= \frac{1}{4}\langle a(t)a(t-\tau)\{\exp(-i\omega\tau)\exp(i[2\omega-2\pi\alpha]t)\\ &\quad + \exp(i\omega\tau)\exp(-i[2\omega+2\pi\alpha]t)\\ &\quad + [\exp(i\omega\tau)+\exp(-i\omega\tau)]\exp(-i2\pi\alpha t)\}\rangle\\ &= \begin{cases}\frac{1}{4}R_{aa}(\tau)\exp(\mp i\omega\tau), & \alpha = \pm\omega/\pi\\ \frac{1}{2}R_{aa}(\tau)\cos(\omega\tau), & \alpha = 0.\end{cases}\end{aligned} \tag{20}$$

The last equality follows from the fact that $a(t)a(t-\tau)$ contains no finite additive sine-wave components and, therefore, $\langle a(t)a(t-\tau)\exp(-i\gamma t)\rangle \equiv 0$ for all $\gamma \neq 0$. Fourier transforming both sides of (20) yields

$$S_{zz}(f;\alpha) = \begin{cases}\frac{1}{4}S_{aa}(f \pm \omega/2\pi), & \alpha = \pm\omega/\pi\\ \frac{1}{4}S_{aa}(f+\omega/2\pi) + \frac{1}{4}S_{aa}(f-\omega/2\pi), & \alpha = 0.\end{cases} \tag{21}$$

For instance, if

$$S_{zz}(f;0) \equiv S_{zz}(f) = \begin{cases}S_0, & ||f| - \omega/2\pi| < B\\ 0, & \text{otherwise,}\end{cases}$$

then

$$S_{zz}(f-\alpha;\alpha) = \begin{cases}S_0, & |f \mp \omega/2\pi| < B, \quad \alpha = \pm\omega/\pi\\ 0, & \text{otherwise.}\end{cases}$$

Hence, if the passband of $H_*(f)$ is covered by the support of $S_{zz}(f)$, as required in Wiener's method (9), then it is also covered by the support of $S_{zz}(f-\alpha;\alpha)$, as required in the SPECCORR method (18).

IV. Application to Interference-Tolerant TDOA Estimation

We consider the signals $x(t)$ and $y(t)$ received by a pair of sensors, such as radio antennas, on which plane waves are impinging from several directions. We let $z(t)$ be the component of $x(t)$ due to a particular plane wave of interest, and we let $w(t)$ be the component of $y(t)$ corresponding to this same plane wave. We then define $n(t) \triangleq x(t) - z(t)$ and $m(t) \triangleq y(t) - w(t)$ to be measurement corruption due to the other plane waves and sensor noise.

Since the only difference between the signals $z(t)$ and $w(t)$ due to a single plane wave is their difference in times of arrival at the two sensors, we have the relationship $w(t) = z(t - t_0)$. Consequently, we have the model (5), (6) of corrupted input/output measurements of an unknown pure delay system with impulse response function $h_*(t) = \delta(t - t_0)$ and transfer function $H_*(f) = \exp(-i2\pi f t_0)$.

If the plane wave of interest exhibits cyclostationarity, as it typically would if it were a radio, radar, or telemetry signal [1], [3], [5], then we can apply the SPECCORR method of system identification to estimate the unknown TDOA (Time Difference of Arrival). From such an estimate, the direction of arrival of the wavefront can be estimated.

Both Wiener's optimum modeling method (9) and the SPECCORR method (18) have been applied to this TDOA estimation problem and extensive simulations have been carried out. The results show that the effects of correlated input/output measurement corruption (interfering plane waves in this case) that are quite problematic for Wiener's method can be essentially eliminated with the SPECCORR method [3], [8]. One such simulation is reported here.

To demonstrate the tolerance to noise and interference exhibited by the SPECCORR method, we consider a BPSK (Binary Phase-Shift Keyed) signal of interest corrupted by multiple interference. Uncorrelated broadband noises are also added to the two received signals. The BPSK signal has carrier frequency of $f_0 = 0.25/T_s$ and baud rate of $\alpha_0 = 0.0625/T_s$. It has full-duty-cycle half-cosine envelope, which results in an approximate bandwidth of $B_0 = 0.1875/T_s$. The TDOA for the signal of interest is $\tau_0 = 48T_s$. The length of the segment of data processed to obtain the TDOA estimates is 2048 keying intervals. T_s is the sampling time in-

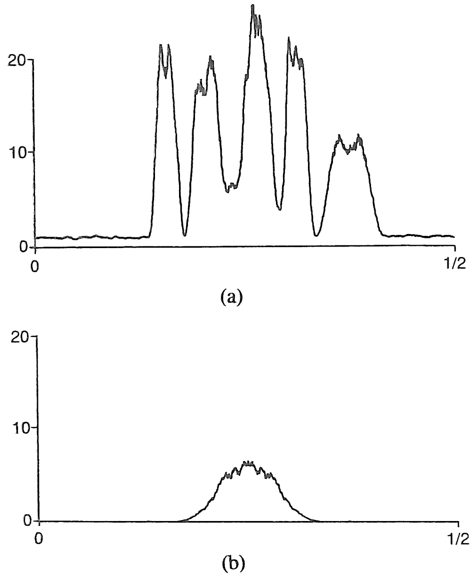

Fig. 1. (a) Graph of the measured spectrum of the corrupted signal $x(t) = z(t)+n(t)$. (b) Graph of the measured spectrum of the uncorrupted signal $z(t)$.

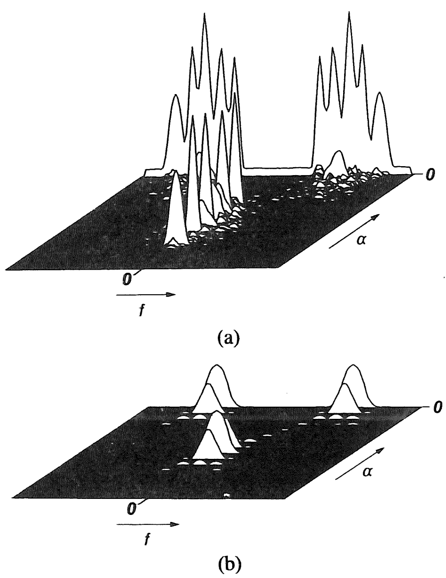

Fig. 2. (a) Graph of the measured cross-spectrum magnitude $|S_{xx}^{\alpha}(f)|$ for the corrupted signal $x(t) = z(t) + n(t)$. (b) Graph of the measured cross-spectrum magnitude $|S_{zz}^{\alpha}(f)|$ for the uncorrupted signal $z(t)$.

crement. Five amptitude-modulated (AM) sine-wave signals are used as interferers. The carrier frequencies are $f_1 = 0.156/T_s$, $f_2 = 0.203/T_s$, $f_3 = 0.266/T_s$, $f_4 = 0.313/T_s$, $f_5 = 0.375/T_s$. The bandwidths are $B_1 = 0.04/T_s$, $B_2 = 0.05/T_s$, $B_3 = 0.045/T_s$, $B_4 = 0.04/T_s$, $B_5 = 0.08/T_s$. The TDOA's are $\tau_1 = 28T_s$, $\tau_2 = 68T_s$, $\tau_3 = 78T_s$, $\tau_4 = 38T_s$, $\tau_5 = 58T_s$. The SIR (Signal-to-Interference Ratio) of each AM Signal is 0 dB and the SNR is 0 dB. Thus, the total SINR is -8 dB. The spectra for this highly corrupted signal $x(t)$ and the uncorrupted signal $z(t)$ are shown in Fig. 1. Although the interfereing AM signals all exhibit cyclostationarity, none of them shares the same cycle frequencies with the BPSK signal. This is illustrated in Fig. 2, which shows a graph of the magnitude of the symmetrized cross spectrum

$$S_{xx}^{\alpha}(f) \triangleq S_{xx}(f - \alpha/2; \alpha), \tag{22}$$

which is the Fourier transform of the symmetrized cross correlation

$$R_{xx}^{\alpha}(\tau) \triangleq \langle x(t + \tau/2) \exp(-i\pi\alpha[t + \tau/2]) \cdot [x(t - \tau/2) \exp(i\pi\alpha[t - \tau/2])]^* \rangle. \tag{23}$$

The cross spectra in the SPECCORR formula (18) were estimated using the conventional method of frequency smoothing cross periodograms obtained from FFT's of the two data records for $x(t)$ and $y(t)$. 32 768 time samples were used for each of the two records, and the number of frequency bins added together in the smoothing operation was 200. This large amount of statistical averaging is required because the corruption is 8 dB stronger than the signal of interest.

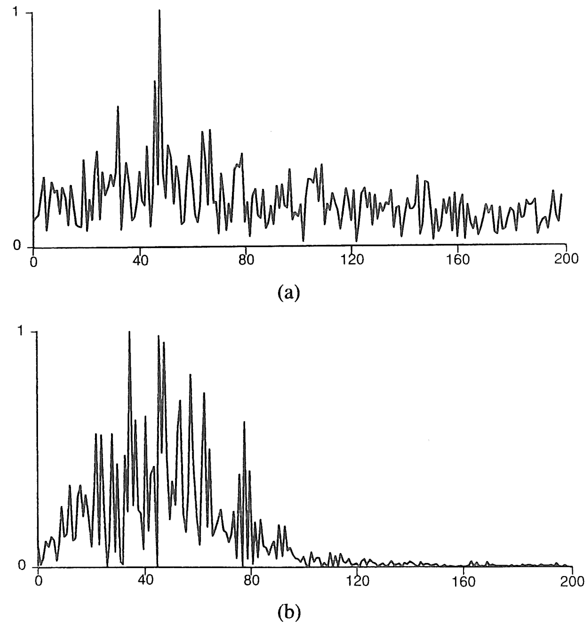

Fig. 3. (a) Graph of the impulse-response estimate obtained with the SPECCORR method. (b) Graph of the impulse-response estimate obtained with Wiener's method.

The ratio of cross-spectrum estimates was processed by an inverse FFT to obtain the final estimate of the impulse response function $h_*(t)$. However, before forming the ratio to be inverse transformed, the denominator values were truncated so that no excessively small values in the denominator would cause numerical instability. The truncation level was chosen to be that level which was exceeded by 95% of the values. The results for the SPECCORR method ($\alpha = \alpha_0$) and Wiener's method ($\alpha = 0$) are shown in Fig. 3. It can be seen that the desired single peak at the TDOA of $48T_s$ is clearly visible for the SPECCORR method, but is severely masked by the interference for Wiener's method.

V. Conclusion

The new SPECCORR method of system identification introduced in this note is quite straightforward: when the uncorrupted input to an unknown system exhibits correlation with frequency shifted versions of itself, by virtue of its cyclostationarity, frequency shifting operations can be used to decorrelate noncyclostationary input/output corruption while maintaining correlation between the shifted input and the system's output to the unshifted input. By this means we are able, in principle, to perfectly identify an unknown system using severely corrupted input/output measurements.

An application to the problem of TDOA estimation demonstrates the effectiveness of the SPECCORR method for system identification with severely corrupted input/output measurements.

References

[1] W. A. Gardner and L. E. Franks, "Characterization of cyclostationary random signal processes," *IEEE Trans. Inform. Theory*, vol. IT-21, pp. 4–14, 1975.
[2] W. A. Gardner, "Stationarizable random processes," *IEEE Trans. Inform. Theory*, vol. IT-24, pp. 8–22, 1978.
[3] ——, *Introduction to Random Processes with Applications to Signals and Systems*, 2nd ed. New York: McGraw-Hill, 1990.
[4] ——, "The spectral correlation theory of cyclostationary time-series," *Signal Processing*, vol. 11, pp. 13–36, 1986.
[5] ——, *Statistical Spectral Analysis: A Nonprobabilistic Theory*. Englewood Cliffs, NJ: Prentice-Hall, 1987.
[6] N. Wiener, *Extrapolation, Interpolation, and Smoothing of Stationary Time Series*. New York: The Technology Press of M.I.T. and Wiley, 1949.
[7] W. A. Gardner, "Signal interception: A unifying theoretical framework for feature detection," *IEEE Trans. Commun.*, vol. 36, pp. 897–906, 1988.
[8] W. A. Gardner and C. K. Chen, "Interference-tolerant time-difference-of-arrival estimation for modulated signals," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 36, pp. 1385–1395, 1988.