Characterization of Cyclostationary Random Signal Processes

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Abstract—**Many communication and control systems employ signal formats that involve some form of periodic processing operation. Signals produced by samplers, scanners, multiplexors, and modulators are familiar examples. Often these signals are appropriately modeled by random processes that are cyclostationary (CS), i.e., processes with statistical parameters, such as mean and autocorrelation, that fluctuate periodically with time. In this paper we examine two methods for representing nonstationary processes that reveal the special properties possessed by CS processes. These representations are the harmonic series representation (HSR) and the translation series representation (TSR). We show that the HSR is particularly appropriate for characterizing the structural properties of CS processes and that the TSR provides natural models for many types of communication signal formats. The advantages gained by modeling signals as CS processes rather than stationary processes is illustrated by consideration of the optimum filtering problem. We present general solutions for filters that minimize mean-square error for continuous-waveform estimation, and we discuss several specific examples for the particular case of additive noise. These examples demonstrate improvement in performance over that of filter designs based on stationary models for the signal processes.**

I. Introduction and Background

RANDOM SIGNAL processes that have been subjected to some form of repetitive operation such as sampling, scanning, or multiplexing will usually exhibit statistical parameters that vary periodically with time. In many cases, the repetitive operation is introduced intentionally to put the signal in a format that is easily manipulated and preserves the time-position integrity of the events that the signal is representing. Familiar examples are radar antenna scanning patterns, raster formats for scanning video fields, synchronous multiplexing schemes, and synchronizing and framing techniques employed in data transmission [1]. In fact, in all forms of data transmission, it seems that some form of periodicity is imposed on the signal format. In contrast to these examples, there are many where the periodicity is not imposed but occurs naturally [2]–[9]. These "cyclostationary" (CS) random processes occur in a wide variety of systems including biological, social, economic, and mechanical as well as electrical systems. They are encountered in studies concerned with physics, meteorology, astronomy, and various other physical and natural sciences.

A continuous-time second-order random process $\{x(t); t \in (-\infty,\infty)\}$ is defined to be CS in the wide sense with period T (which we shall hereafter abbreviate as CS(T)), if and only if its mean and autocorrelation exhibit the periodicity [1]

$$m_x(t) \triangleq E\{x(t)\} = m_x(t+T), \qquad \text{for all } t \in (-\infty,\infty)$$

$$k_{xx}(t,s) \triangleq E\{x(t)x^*(s)\} = k_{xx}(t+T, s+T), \qquad \text{for all } t, s \in (-\infty,\infty). \quad (1\text{-}1)$$

Bennett [10] introduced the term "cyclostationary" to denote this class of processes in his treatment of synchronously timed pulse sequences used in digital data transmission. Other investigators [5]–[12] have used terms such as "periodically stationary," "periodically correlated," and "periodic nonstationary" to denote this same class.

A practical example of a CS signal process is the pulse-amplitude-modulated (PAM) signal

$$x(t) = \sum_{n=-\infty}^{\infty} a_n \phi(t - nT) \quad (1\text{-}2)$$

where ϕ is a deterministic $L^2(-\infty,\infty)$ function. The mean and autocorrelation for this signal are easily shown to satisfy (1-1) if the random amplitude sequence $\{a_n\}$ is wide-sense stationary (WSS) [1]. In many applications, the pulse amplitudes are obtained by uniformly time-sampling a WSS process. Other forms of synchronous pulse-modulated signals—including frequency- and phase-shift keyed signals (FSK) and (PSK), respectively, and pulse-width and pulse-position modulated signals (PWM) and (PPM), respectively—are also CS when modeled as

$$x(t) = \sum_{n=-\infty}^{\infty} \phi(t - nT, a_n) \quad (1\text{-}3)$$

provided that the modulating sequence $\{a_n\}$ is stationary of order two. In fact, even asynchronous pulse sequences that result from including random epoch jitter in the preceding model, by replacing nT with $nT + \delta_n$, are CS (T) if the jitter sequence $\{\delta_n\}$ is jointly (with $\{a_n\}$) stationary of order two.

Another example is the totally asynchronous pulse sequence that results from replacing the deterministic epochs $\{nT\}$ in (1-2) with the random epochs $\{t_n\}$, which form an ordered sequence distributed according to the nonhomogeneous Poisson counting process. This model has been used for telegraph and facsimile signals and noise in electronic devices [1]. If the counting rate parameter fluctuates periodically, then the process is CS. There are numerous other signal formats used in communication and control that result in CS processes. These include amplitude-modulated signals (AM), analog and digital phase- and frequency-modulated signals (PM) and (FM), respectively,

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time- and frequency-division multiplexed signals (TDM) and (FDM), respectively, and video signals.[1]

Systems analysts have, for the most part, treated these CS signals as though they were stationary. This is done simply by averaging the statistical parameters over one cycle

$$\tilde{m}_x = \frac{1}{T}\int_{-T/2}^{T/2} m_x(t)\,dt$$

$$\tilde{k}_{xx}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} k_{xx}(t+\tau, t)\,dt. \tag{1-4}$$

This averaging is equivalent to modeling the time reference or phase of the process as a random variable uniformly distributed over one cycle

$$\tilde{x}(t) = x(t+\theta)$$

$$p_\theta(\alpha) = \begin{cases} \frac{1}{T}, & |\alpha| \le \frac{T}{2} \\ 0, & |\alpha| > \frac{T}{2} \end{cases} \tag{1-5}$$

where p_θ is the probability density function for θ [14]. In this case, the mean and autocorrelation for the phase-randomized process $\tilde{x}$ are given by the time-averaged functions in (1-4). This type of analysis may be appropriate in situations where the signal is not observed in synchronism with its periodic structure. For example, a CS process may be an interference in another signal-transmission channel with a receiver that has no knowledge of the phase of the interfering process. In such a situation, the concepts used with stationary processes, such as power spectral density, can be valuable in evaluating system performance. On the other hand, in a receiver that is intended for the CS process there is usually a great deal of information provided, in the form of synchronizing pulses or a sinusoidal timing signal, about the exact phase of the signal format. Most of these systems are, in fact, inoperative without this information. In these situations, the CS model is more appropriate.

One way that the CS model for a signal process can be used to advantage is in the design of optimum filters. Previous analyses of optimum filtering operations have usually assumed the stationary model for signal processes and result in time-invariant filters that "ignore" the cyclic fluctuations in these processes. We shall demonstrate in Section III that improved performance can be obtained by recognizing that—by virtue of the timing information at the receiver—the received process is actually CS, and the optimum filter is a synchronized periodically time-varying filter.

The CS model is also useful in the design and analysis of synchronization schemes for extracting timing information from received CS signals [16].

Several investigators have recognized the importance of this class of nonstationary processes, and they have contributed to its characterization through the development and analysis of representations for CS processes and their autocorrelation functions. Gudzenko [7], Gladyshev [8], and Hurd [5], [15] have contributed to the development and analysis of the following Fourier series representation for periodic autocorrelation functions for harmonizable as well as nonharmonizable CS processes[2]

$$k_{xx}(t,s) = \sum_{n=-\infty}^{\infty} c_n(t-s)\exp\left[\frac{j\pi n(t+s)}{T}\right]$$

$$c_n(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} k_{xx}(s+\tau \mid 2, s-\tau \mid 2)\exp\left(\frac{-j2\pi ns}{T}\right) ds. \tag{1-6}$$

Although this representation does not result from a corresponding representation for the process x, we have derived the following new relationship between the coefficient functions $\{c_n\}$ and the process x

$$c_n(t-s) = E\{\tilde{b}_n(t)\tilde{b}_{-n}{}^*(s)\} \tag{1-7}$$

where $\tilde{b}_n$ is the phase-randomized version of the frequency-translated process b_n

$$\tilde{b}_n(t) \triangleq b_n(t+\theta)$$

$$b_n(t) \triangleq x(t)\exp\left(\frac{-j\pi nt}{T}\right). \tag{1-8}$$

In addition to providing a link between x and $\{c_n\}$, this relationship immediately yields the known result that c_0 is the autocorrelation function for the stationary phase-randomized process $\tilde{x}$.

In spite of this link, the lack of a corresponding process representation appears to be responsible for the apparent lack of useful properties and applications of this Fourier series representation for autocorrelation functions. More recent investigations have concentrated on series representations for CS processes, from which corresponding representations for autocorrelation functions follow.

Jordan, in his work on optimum discrete representations [17], observed that the Karhunen–Loève representation (KLR) for a CS (T) process on the interval $[nT,(n+1)T]$ can be obtained from the KLR on $[0,T]$ simply by employing time-translated versions of the basis functions on $[0,T]$. Doing this for all such intervals results in the mean-square equivalent translation series representation (TSR)

$$x(t) = \sum_{n=-\infty}^{\infty}\sum_{p=1}^{\infty} a_p(n)\phi_p(t-nT), \qquad \text{for all } t \in (-\infty,\infty)$$

$$a_p(n) \triangleq \int_0^T x(t+nT)\phi_p{}^*(t)\,dt \tag{1-9}$$

where $\{\phi_p\}$ are the eigenfunctions (extended to zero outside $[0,T]$) of the linear integral operator on $L^2[0,T]$ with kernel k_{xx}. Similarly, Brelsford [9] observed that a CS (T)

[1] A more detailed discussion of the modeling of the signals mentioned here and various others can be found in [13].

[2] Hurd's work on measurable second-order CS processes is the most comprehensive.

process can be represented on $(-\infty,\infty)$ with a TSR, as in (1-9), by employing any orthogonal decomposition of x on $[0,T]$—the motivation being to decompose continuous-time CS processes into jointly WSS discrete-time processes.

The straightforwardness and simplicity of these TSR's are probably responsible for the apparent lack of further investigations into representations of this type. However, as we shall demonstrate in Sections II and III, these TSR's are very useful for modeling many communication signal formats and solving associated estimation and detection problems. They also provide the first step in the development of other representations.

Another approach to the representation of CS processes was taken by Ogura [6], who derived a spectral representation that leads to the following mean-square equivalent harmonic series representation (HSR) for harmonizable processes

$$x(t) = \sum_{p=-\infty}^{\infty} a_p(t) \exp\left(\frac{j2\pi pt}{T}\right), \qquad \text{for all } t \in (-\infty,\infty)$$

$$a_p(t) \triangleq \int_{-\infty}^{\infty} w(t-\tau)x(\tau) \exp\left(\frac{-j2\pi p\tau}{T}\right) d\tau$$

$$w(t) \triangleq \frac{\sin(\pi t/T)}{\pi t}. \tag{1-10}$$

Ogura's spectral representation for continuous-time CS processes is complemented by Brelsford's spectral representation for discrete-time CS processes [9].

The first applications of the HSR and the TSR to the characterization of CS processes through the identification of various properties were presented in our preliminary paper [18]. It is our intent in this paper to present a more complete account of our findings on the use of series representations for characterizing CS processes.[3] Although our development and interpretations of the properties of the HSR and TSR go well beyond those of previous investigations, it is not our intent to present these new results with emphasis on generality or rigor. Rather, we have chosen to emphasize their utility, in a somewhat tutorial fashion, with practical examples of applications to optimum filtering problems.

II. Series Representations and Structural Properties

Series representations for continuous-time processes can be categorized into two classes: discrete series representations such as the TSR, wherein the representors are discrete-parameter random processes (random sequences), and continuous-series representations such as the HSR, wherein the representors are continuous-parameter random processes. As discussed in the sequel, the HSR has been most useful for characterizing the structural properties of CS processes, whereas the TSR has been most useful for modeling random signal processes and solving associated detection and estimation problems. However, despite these differences, we shall show that these two types of representation are intimately related.

Harmonic Series Representation

The HSR of (1-10) is most transparent when viewed in the "frequency domain." This representation partitions the frequency support of x into bands of width $1/T$, so that the pth component $a_p(t)\exp(j2\pi pt/T)$ is simply the output of an ideal one-sided bandpass filter with input x and transfer function

$$W_p(f) = \begin{cases} 1, & \left|f - \dfrac{p}{T}\right| \le \dfrac{1}{2T}; \\ 0, & \text{otherwise.} \end{cases}$$

The representor $a_p(\cdot)$ is the centered (to the frequency-band $[-1/2T, 1/2T]$) version of the pth component $a_p(\cdot)\exp[j2\pi p(\cdot)/T]$.

Although the HSR is a valid mean-square equivalent representation for noncyclic nonstationary harmonizable processes, it is particularly appropriate for CS processes since it accomplishes a decomposition of CS (T) processes into a countable set of jointly WSS bandlimited processes, viz., the representors $\{a_p\}$, which are jointly WSS if and only if x is CS (T)[4] [18].

The autocorrelation function for a harmonizable CS (T) process has a harmonic series representation that corresponds to the HSR for the process itself

$$k_{xx}(t,s) = \sum_{p,q=-\infty}^{\infty} r_{pq}(t-s) \exp\left[\frac{j2\pi(pt-qs)}{T}\right], \qquad \text{for all } t, s \in (-\infty,\infty)$$

$$\begin{aligned} r_{pq}(t-s) &\triangleq E\{a_p(t)a_q^*(s)\} \\ &= \iint_{-\infty}^{\infty} w(t-\tau)w(s-\gamma)k_{xx}(\tau,\gamma) \cdot \exp\left[\frac{-j2\pi(p\tau - q\gamma)}{T}\right] d\tau\, d\gamma \\ &= \iint_{-\infty}^{\infty} W_0(f)W_0(\nu)\exp[j2\pi(ft-\nu s)] \, K_{xx}(f+p/T, \nu+q/T)\, df\, d\nu \\ &= \iint_{-1/2T}^{1/2T} K_{xx}(f+p/T, \nu+q/T) \cdot \exp[j2\pi f(t-s)]\, df\, d\nu \end{aligned} \tag{2-1}$$

where K_{xx} is the double Fourier transform[5] of k_{xx}. This last equality follows from the fact that $K_{xx}(f,\nu)$ consists of impulse fences on lines parallel to the $f = \nu$ diagonal and separated by $1/T$, as shown in the sequel. From this last

[3] This paper is a summary of parts of the unpublished report [13].

[4] This is an important result that was not stated (or inferred) in an earlier work [6].

[5] We are employing the extended Fourier transform that is defined for periodic functions and may include impulse functions [11].

equality, we see that the cross spectral densities $\{R_{pq}\}$ for the $\{a_p\}$ are related to the bifrequency spectrum K_{xx} as follows:

$$R_{pq}(f) \triangleq \int_{-\infty}^{\infty} r_{pq}(\tau) \exp(-j2\pi f\tau)\, d\tau$$

$$= W_0(f) \int_{-1/2T}^{1/2T} K_{xx}(f + p \mid T, \nu + q \mid T)\, d\nu$$

and

$$K_{xx}(f,\nu) = \sum_{p,q} R_{pq}(f - p \mid T)\, \delta(f - \nu + (q + p) \mid T). \quad (2\text{-}2)$$

Note that

$$\int_{-\infty}^{\infty} \int_0^T |k_{xx}(t + \tau, t)|^2\, dt\, d\tau < \infty \Rightarrow r_{pq}(\cdot) \in L^2(-\infty,\infty).$$

In fact, since the spectral densities $\{R_{pq}\}$ are related to the Fourier transforms of the coefficient function $\{c_n(\cdot)\}$ in the Fourier series representation (1-6) by the simple relation

$$R_{pq}(f) = C_{p-q}(f + (p + q) \mid 2T)\, W_0(f) \quad (2\text{-}3)$$

then various properties of $\{R_{pq}\}$ and conditions for—and modes of—convergence of (2-1) can be obtained from Hurd's extensive results for $\{C_n\}$ and (1-6) [5], [15].

The utility of the HSR for characterizing CS processes is illustrated by the following properties of the spectral density matrix $\boldsymbol{R}(f)$ with elements $\{R_{pq}(f)\}$.

1) *Stationarity:* Since the components

$$\left\{a_p(t) \exp\left(\frac{j2\pi pt}{T}\right)\right\}$$

of the HSR are individually WSS but are not jointly WSS (because of the exponential factors), then a zero-mean CS process will generally be WSS, if and only if the representors $\{a_p\}$ are uncorrelated, i.e., if and only if the spectral density matrix $\boldsymbol{R}$ is diagonal.

2) *Phase Randomization:* If $\tilde{x}$ is the process that is derived from a CS (T) process x by the introduction of a random phase variable θ, $\tilde{x}(t) \triangleq x(t + \theta)$, then $\tilde{x}$ is also CS (T) [14]. Furthermore, if $p_\theta(\cdot)$ is the probability density function for θ and $P_\theta(\cdot)$ is the Fourier transform (conjugate characteristic function), then the elements of the spectral density matrix $\tilde{\boldsymbol{R}}(f)$ for $\tilde{x}$ can be shown to be related to the elements of the matrix $\boldsymbol{R}(f)$ for x by the formula

$$\tilde{R}_{pq}(f) = P_\theta\left(\frac{p - q}{T}\right) R_{pq}(f). \quad (2\text{-}4)$$

Now, since $p_\theta(\cdot)$ is a probability density function, then $P_\theta(n/T) \le P_\theta(0) = 1$, and we see that the off-diagonal elements of $\boldsymbol{R}$ are attenuated by the phase randomization. Note that $\tilde{\boldsymbol{R}}$ will be diagonal and $\tilde{x}$ WSS, for every x, if and only if $P_\theta[(p - q)/T] = \delta_{pq}$ (the Kronecker delta), which is satisfied by a uniform density function. Furthermore, the power spectral density for this stationarized process is given by the formula

$$\tilde{K}_{xx}(f) = \sum_{p=-\infty}^{\infty} R_{pp}(f - p \mid T). \quad (2\text{-}5)$$

3) *Time-Invariant Filtering:* If $\tilde{x}$ is the output of a time-invariant filter with CS (T) input x, then $\tilde{x}$ is also CS (T). Furthermore, if the transfer function for the filter is denoted $G(\cdot)$, then the elements of the spectral density matrix $\tilde{\boldsymbol{R}}$ for $\tilde{x}$ can be related to the elements of the matrix $\boldsymbol{R}$ for x by the formula

$$\tilde{R}_{pq}(f) = G(f + p \mid T)\, G^*(f + q \mid T)\, R_{pq}(f). \quad (2\text{-}6)$$

Note that if the filter is ideal low-pass with cutoff frequency B, then all elements in $\tilde{\boldsymbol{R}}$ with indices $|p|$ or $|q| > BT$ are identically zero.

4) *Bandlimitedness:* A random process is said to be bandlimited to the band $[-B,B]$ if it is unaffected by passage through an ideal low-pass filter with cutoff frequency B. In this case, the double Fourier transform of its autocorrelation function satisfies the bandlimiting condition $K_{xx}(f,\nu) = 0$, for $|f| > B$ and for $|\nu| > B$. Thus, using either (2-2) or (2-6), we see that if x is CS (T) and bandlimited to $[-B,B]$, then it admits an Mth order HSR, where $M \le BT$, i.e., the HSR representors $\{a_p;\ |p| > M\}$ are identically zero. Hence, if $B \le 1/2T$, then x is WSS.

5) *Degree of CS:* The relative magnitudes of the off-diagonal elements in the spectral matrix $\boldsymbol{R}$ appear to be a useful indication of the degree of CS of a process. For example, phase randomization tends to reduce the degree of CS, and it attenuates the off-diagonal elements. In fact, when the random phase is uniformly distributed, then the off-diagonal elements are identically zero, and the process is stationary (zero degree of cyclostationarity). Furthermore, low-pass filtering tends to reduce the degree of CS, and it reduces the size of higher order off-diagonal elements. In fact, when the bandwidth is strictly limited to $1/2T$, then all off-diagonal elements are zero, and the process is again stationary. A convenient measure of the degree of CS would be very useful in analyses dealing with CS processes, as shown in the following discussion of optimum filtering and as discussed in another paper [16] in relation to timing extraction.

Translation Series Representation

The TSR of (1-9), where the $\{\phi_p\}$ comprise any complete orthonormal set on $L^2[0,T]$ (extended to the zero function outside $[0,T]$), is a valid mean-square equivalent representation for noncyclic nonstationary mean-square continuous processes. However, it is particularly appropriate for CS (T) processes, since in this case it decomposes the continuous-time process x into jointly WSS random sequences. That is, the discrete representors $\{\{a_p(\cdot)\};\ p = 1,2,\cdots\}$ are jointly WSS, if and only if x is CS (T). The autocorrelation function for a CS (T) process has a translation series representation that corresponds to the TSR for the process itself

$$k_{xx}(t,s) = \sum_{n,m=-\infty}^{\infty} \sum_{p,q=1}^{\infty} r_{pq}(n - m)\phi_p(t - nT)\phi_q^*(s - mT),$$

$$\text{for all } t, s \in (-\infty,\infty)$$

and

$$r_{pq}(n-m) \triangleq E\{a_p(n)a_q^*(m)\} = \int_0^T \int_0^T k_{xx}(t+(n-m)T, s)\phi_p^*(t)\phi_q(s)\,dt\,ds. \tag{2-7}$$

From (2-7), the double Fourier transform of k_{xx} is

$$K_{xx}(f,v) = \frac{1}{T}\sum_{p,q=1}^{\infty} R_{pq}(f)\Phi_p(f)\Phi_q^*(v) \sum_{n=-\infty}^{\infty} \delta(f-v+n\mid T) \tag{2-8}$$

where the $\{R_{pq}\}$ are the elements of the spectral density matrix $R(f)$

$$R_{pq}(f) \triangleq \sum_{n=-\infty}^{\infty} r_{pq}(n)\exp(-j2\pi nTf). \tag{2-9}$$

Note that if

$$\int_{-\infty}^{\infty}\int_0^T |k_{xx}(t+\tau,t)|^2\,dt\,d\tau < \infty$$

then the sequences $\{r_{pq}(n)\}$ are square-summable.

Observe that K_{xx} consists of impulse fences on lines parallel to the diagonal $f = v$, and separated by $1/T$. If x is stationarized, then only the fence along the line $f = v$ remains, and the power spectral density is given by the quadratic form

$$\tilde{K}_{xx}(f) = \frac{1}{T}\Phi'(f)R(f)\Phi^*(f). \tag{2-10}$$

Another class of discrete series representations for nonstationary processes is the time-frequency dual of the TSR

$$x(t) = \sum_p\sum_n b_n(p)\theta_n(t)\exp\left(\frac{j2\pi pt}{T}\right), \quad \text{for all } t\in(-\infty,\infty)$$

$$b_n(p) \triangleq \int_{-\infty}^{\infty} x(t)\theta_n^*(t)\exp\left(\frac{-j2\pi pt}{T}\right)dt \tag{2-11}$$

where the Fourier transforms $\{\Theta_n\}$ of the basis functions comprise any complete orthonormal set on $L^2[-1/2T, 1/2T]$ (extended to zero outside $[-1/2T, 1/2T]$). The basis functions in this frequency TSR are translated in frequency—by means of the complex exponential factors—by integer multiples of $1/T$, whereas the basis functions in the time TSR (1-9) are translated in time by integer multiples of T. Note that $\{b_n(p)\}$ is a discrete representation (with respect to the basis $\{\theta_n\}$) for the HSR representor a_p.

Note also that with the particular basis

$$\theta_n(t) = \sqrt{T}\,\frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)}$$

$$\Theta_n(f) = \begin{cases} \sqrt{T}\exp(-j2\pi nfT), & |f| \le \dfrac{1}{2T} \\ 0, & |f| > \dfrac{1}{2T} \end{cases} \tag{2-12}$$

(2-11) is a TSR in both the time and frequency domains; however, the basis functions in the time-domain TSR are not duration-limited to $[0,T]$

$$\phi_p(t) = \sqrt{T}\exp\left(\frac{j2\pi pt}{T}\right)\frac{\sin(\pi t/T)}{\pi t}$$

$$a_p(n) = b_n(p).$$

Furthermore, the spectral density matrix for this TSR is simply the periodic $(1/T)$ extension of the spectral density matrix for the HSR.

As a final note, we mention that with the nonduration-limited basis functions

$$\phi_p(t) = \left(\frac{M}{T}\right)^{1/2}\frac{\sin[(t-pT/M)\pi M/T]}{(t-pT/M)\pi M/T} \tag{2-13}$$

which are orthonormal on $L^2(-\infty,\infty)$, the TSR of (1-9)—with the interval of integration changed from $[0,T]$ to $(-\infty,\infty)$, and the range of summation over p reduced from $[1,\infty)$ to $[1,M]$—is in fact the sampling representation for nonstationary processes bandlimited to the frequency-interval $[-M/2T, M/2T]$ [19].

In the next section, we demonstrate the practical utility of the TSR for modeling important types of communication signal formats and for solving optimum filtering problems.

III. Continuous-Waveform Estimation

In this section we consider the problem of minimum-mean-square error (MMSE) estimation of CS processes. In particular, we consider the problem of linear noncausal continuous-waveform estimation (optimum noncausal filtering). We restrict our attention here to *noncausal* filtering because the results are easier to interpret, and because these results provide a lower bound on mean-square error that, by incorporating a sufficient time delay in the estimate, can be approached by a causal filter [20].

The optimum noncausal filter is the linear system with impulse-response function $h(\cdot,\cdot)$ that minimizes the mean-square error

$$J(t) \triangleq E\{[x(t)-\hat{x}(t)]^2\}$$

$$\hat{x}(t) = \int_{-\infty}^{\infty} h(t,s)y(s)\,ds \tag{3-1}$$

for every $t\in(-\infty,\infty)$, where x is the random process to be estimated (transmitted signal), and y is the observed process (received signal) from which the estimate (filtered signal) $\hat{x}$ is to be obtained.

The following linear integral equation is well known to be the necessary and sufficient condition that implicitly specifies the impulse-response function for the optimum filter [1]

$$\int_{-\infty}^{\infty} h(t,\sigma)k_{yy}(\sigma,s)\,d\sigma = k_{xy}(t,s), \quad \text{for all } t, s\in(-\infty,\infty) \tag{3-2}$$

where k_{yy} is the autocorrelation function for y and k_{yx} is the cross correlation for y and x. The minimum estimation error that remains after optimum filtering is given by the formula [1]

$$J_0(t) = k_{xx}(t,t) - \int_{-\infty}^{\infty} h(t,s)k_{xy}(t,s)\,ds, \quad \text{for all } t \in (-\infty,\infty). \tag{3-3}$$

If y and x are jointly CS (T), then the optimum filter is a periodically (T) time-varying system and the minimum estimation error is a periodic (T) function of time.

We shall solve (3-2) for this optimum time-varying filter, and evaluate the time-averaged value of the minimum estimation error,

$$\langle J_0 \rangle \triangleq \frac{1}{T}\int_0^T J_0(t)\,dt. \tag{3-4}$$

We shall also solve for the time-invariant filter that minimizes the time-averaged value of the estimation error $\langle J \rangle$, evaluate the minimum value of this time-averaged error $\langle J \rangle_0$, and evaluate the relative performance measure (improvement factor)

$$P \triangleq \frac{\langle J \rangle_0}{\langle J_0 \rangle} \geq 1. \tag{3-5}$$

First, we state the following new and important theorem on the equivalence of optimum time-invariant filters for CS processes.

Theorem 1: The optimum filter for the stationarized (phase-randomized) versions $\tilde{x}, \tilde{y}$ of the CS processes x,y is identical to the time-invariant filter that minimizes the time-averaged value of the periodic mean-square estimation error $\langle J \rangle$ for x,y, and it has transfer function

$$\tilde{H}(f) = \frac{\tilde{K}_{xy}(f)}{\tilde{K}_{yy}(f)}. \tag{3-6}$$

The minimum time-averaged estimation error is given by the formula

$$\langle J \rangle_0 = \tilde{k}_{xx}(0) - \int_{-\infty}^{\infty} \frac{|\tilde{K}_{xy}(f)|^2}{\tilde{K}_{yy}(f)}\,df. \tag{3-7}$$

In contrast to the explicit solution of Theorem 1 for optimum time-invariant filters, there is no previously known generally applicable solution for optimum time-varying filters for CS processes. In this section we employ the series representations of Section II to obtain general solutions, and we illustrate these solutions with specific examples.

Theorem 2: The impulse-response function h for the optimum time-varying filter for the general MMSE estimation problem (the solution to (3-2)) admits the TSR

$$h(t,s) = \sum_{n,m=-\infty}^{\infty} \sum_{p,q=1}^{M} h_{pq}(n-m)\phi_p(t-nT)\phi_q^*(s-mT), \quad \text{for all } t,s \in [-\infty,\infty] \tag{3-8}$$

provided that

1) the transmitted and received signals x,y are jointly CS (T);
2) y is composed of the sum of a colored component z and a white component, with power spectral density λ, that is uncorrelated with x and z;
3) x and z admit Mth order TSR's with orthonormal basis $\{\phi_p\}$ on $[0,T]$.

The sequence of matrices $\{\boldsymbol{h}(n)\}$ is given by the inverse z transform

$$\boldsymbol{h}(n) = T\int_{-1/2T}^{1/2T} \boldsymbol{H}(f)\exp(j2\pi nTf)\,df \tag{3-9}$$

where the matrix of functions $\boldsymbol{H}(f)$ is given by the formula

$$\boldsymbol{H}(f) = \boldsymbol{C}(f)[\boldsymbol{D}(f) + \lambda \boldsymbol{I}]^{-1}. \tag{3-10}$$

The matrices $\boldsymbol{D}, \boldsymbol{C}$ are spectral densities [z transforms evaluated at $z = \exp(-j2\pi fT)$, as defined by (2-9)] corresponding to the correlation matrices

$$d_{pq}(n) = \iint_0^T k_{zz}(t+nT, s)\phi_p^*(t)\phi_q(s)\,dt\,ds$$

$$c_{pq}(n) = \iint_0^T k_{xz}(t+nT, s)\phi_p^*(t)\phi_q(s)\,dt\,ds \tag{3-11}$$

and $\boldsymbol{I}$ is the identity matrix. Furthermore, the time-averaged value of the minimum estimation error is given by the formula

$$\langle J_0 \rangle = \int_{-1/2T}^{1/2T} \operatorname{tr}[\boldsymbol{R}(f) - \boldsymbol{H}(f)\boldsymbol{C}'(f)^*]\,df \tag{3-12}$$

where $\boldsymbol{R}(f)$ is given by (2-9).

Proof: Substituting the TSR's for k_{zz}, k_{xz}, h into the necessary and sufficient condition (3-2) yields the following necessary and sufficient condition on $\{h_{pq}(n-m)\}$

$$\sum_{i=1}^{M} \sum_{m=-\infty}^{\infty} h_{pi}(n-m)d_{iq}(m) + \lambda h_{pq}(n) = c_{pq}(n), \quad \text{for all } n, \text{ and all } |p|, |q| \leq M. \tag{3-13}$$

Recognizing this double sum as the composition of a discrete convolution and a matrix product, and equating z transforms of both sides of the equation yields (from the convolution theorem) the necessary and sufficient condition on $\boldsymbol{H}(f)$

$$\boldsymbol{H}(f)\boldsymbol{D}(f) + \lambda\boldsymbol{H}(f) = \boldsymbol{C}(f), \quad \text{for all } f \in [-1/2T, 1/2T] \tag{3-14}$$

which is satisfied by (3-10). Now the formula (3-12) for $\langle J_0 \rangle$ can be verified by substituting the TSR's into (3-3) and performing the integration indicated in (3-4) to obtain the formula

$$\langle J_0 \rangle = \frac{1}{T}\sum_{p=1}^{M}\left[r_{pp}(0) - \sum_{m=-\infty}^{\infty}\sum_{q=1}^{M} h_{pq}(m)c_{pq}^*(m)\right] \tag{3-15}$$

$$= \int_{-1/2T}^{1/2T} \operatorname{tr}[\boldsymbol{R}(f) - \boldsymbol{H}(f)\boldsymbol{C}'(f)^*]\,df. \tag{3-16}$$

Q.E.D.

The structure of the optimum time-varying filter of this theorem is shown in Fig. 1, and it consists of an input bank of time-invariant filters with outputs that are periodically (T) sampled to yield sequences that are applied to a multi-

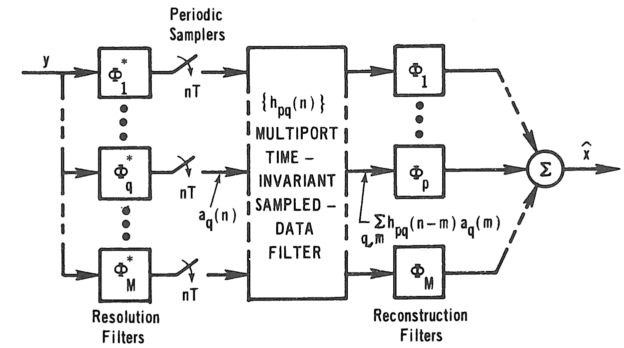

Fig. 1. Structure of optimum periodically time-varying filter with impulse-response function that admits an Mth order TSR (Theorem 2). $\{a_q(n)\}$ are the TSR representors for the received signal y.

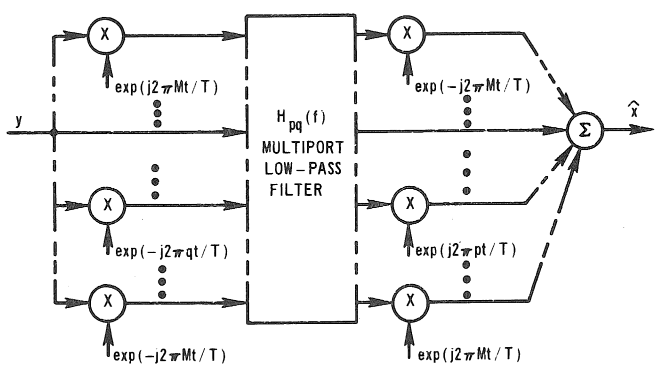

Fig. 2. Structure of optimum periodically time-varying filter with impulse-response function that admits an Mth order HSR (Theorem 3). $H_{pq}(f) = 0$, for $|f| \geq 1/2T$.

port time-invariant sampled-data filter; the filtered sequences are then employed to impulse-excite an output bank of time-invariant filters with outputs that are summed to form the MMSE continuous-waveform estimate. Notice that the input bank of filters and samplers resolves the received CS signal into WSS TSR representor sequences and that the output bank of filters and the summer reconstructs the optimum signal estimate $\hat{x}$ from the optimally filtered sequences. Hence the optimum filtering is executed by the sampled-data filter, and the remainder of the structure merely accomplishes resolution and reconstruction. By employing TSR's we convert the problem of MMSE estimation of a scalar CS process to the problem of MMSE estimation of a vector of jointly WSS sequences.[6]

The approach of this theorem can be extended to those cases where the processes admit TSR's with basis functions that are not duration limited to $[0,T]$, and/or not orthonormal.

For example, the TSR of (2-11), (2-12) can be used with the extended version of Theorem 2 to derive Theorem 3, which provides a frequency-domain type of solution. Theorem 2 provides a time-domain type of solution.

Theorem 3: The impulse-response function h for the optimum time-varying filter for the general MMSE estimation problem admits the HSR

$$h(t,s) = \sum_{p,q=-M}^{M} h_{pq}(t - s) \exp\left(\frac{j2\pi(pt - qs)}{T}\right), \qquad \text{for all } t, s \in (-\infty,\infty) \quad (3\text{-}17)$$

provided that

1) the transmitted and received signals x,y are jointly CS (T);
2) y is composed of the sum of a colored component z and a white component, with power spectral density λ, that is uncorrelated with x and z;
3) x and z admit Mth order HSR's.

The matrix of Fourier transforms of the elements $\{h_{pq}\}$ is given in (3-10), where the elements of the spectral density matrices $\boldsymbol{D},\boldsymbol{C}$ are given by the formulas

$$D_{pq}(f) = W_0(f) \int_{-1/2T}^{1/2T} k_{zz}\,(f + p \mid T, v + q \mid T)\, dv$$
$$C_{pq}(f) = W_0(f) \int_{-1/2T}^{1/2T} K_{xz}\,(f + p \mid T, v + q \mid T)\, dv. \qquad (3\text{-}18)$$

Furthermore, the time-averaged value of the minimum estimation error is given in (3-12) where $\boldsymbol{R}$ is given by (2-2).

The structure of the optimum time-varying filter of this theorem is shown in Fig. 2 and consists of an input bank of harmonic multipliers with outputs that are applied to a multiport time-invariant low-pass filter. The filtered components are applied to an output bank of harmonic multipliers with outputs that are summed to form the MMSE estimate. Note the parallel to the TSR structure of Fig. 1.[7]

Although Theorems 2 and 3 are stated and proved for finite-order representations (finite M), these solutions do extend to the case where M is infinite for special subclasses of infinite-order representations. This is illustrated in example (5).

Examples

We now discuss various examples of time-varying filters for CS signals in additive noise. We begin with an extended version of Theorem 2 that is particularly convenient for a class of optimum filtering problems frequently encountered in studies of communication systems. Specifically, we assume that a transmitted signal x has been subjected to time-invariant channel dispersion and additive WSS noise, so that the received signal is given by

$$y(t) = \int_{-\infty}^{\infty} g(t - \tau)x(\tau)\, d\tau + n(t), \qquad \text{for all } t \in (-\infty,\infty) \quad (3\text{-}19)$$

where g is the impulse-response function for the dispersion (transfer function $G(f)$) and n is the additive noise (power spectral density $K_{nn}(f)$). We further assume that x admits

[6] This filter is substantially different from that suggested by Jordan which performs independent fixed interval smoothing for each interval $[nT,(n + 1)T]$ [17].

[7] The structure of this filter resembles that of Brelsford's autoregressive predictor for CS sequences [9].

an Mth order TSR with basis functions $\{\phi_p\}$ that are not necessarily duration-limited, nor orthonormal

$$x(t) = \sum_{p=1}^{M} \sum_{n=-\infty}^{\infty} a_p(n)\phi_p(t - nT), \qquad \text{for all } t \in (-\infty,\infty). \tag{3-20}$$

For this class of filtering problems, the optimum time-varying filter is given by the formula

$$h(t,s) = \sum_{p,q} \sum_{n,m} h_{pq}(n - m)\phi_p(t - nT)\theta_q^*(s - mT), \qquad \text{for all } t,\, s \in (-\infty,\infty) \tag{3-21}$$

where $\{h_{pq}\}$ are given by (3-9) with $H(f)$ defined as

$$H(f) \triangleq R(f)[L(f)R(f) + I]^{-1} \tag{3-22}$$

where $R(f)$ is the spectral density matrix for $\{a_p\}$ and $L(f)$ is defined as

$$L(f) \triangleq \sum_n \ell(n) \exp(-j2\pi nTf)$$

$$\ell_{pq}(n) \triangleq \int_{-\infty}^{\infty} \frac{|G(f)|^2\Phi_p^*(f)\Phi_q(f)}{K_{nn}(f)} \exp(j2\pi nTf)\, df. \tag{3-23a}$$

θ_p is the inverse Fourier transform of the function

$$\Theta_p(f) \triangleq \frac{G(f)\Phi_p(f)}{K_{nn}(f)}. \tag{3-24}$$

Furthermore, the time-averaged value of the minimum filtering error is given by the formula

$$\langle J_0 \rangle = \int_{-1/2T}^{1/2T} \operatorname{tr}[B(f)H(f)]\, df \tag{3-25}$$

where $B(f)$ is $L(f)$ with $|G(f)|$ and $K_{nn}(f)$ replaced by 1; i.e.,

$$b_{pq}(n) = \int_{-\infty}^{\infty} \phi_p^*(t - nT)\phi_q(t)\, dt. \tag{3-26}$$

This solution is easily verified using a procedure paralleling that given in the appendix of [21].

Note that the Poisson sum formula [1] can be employed to convert (3-23a) into the alternate form

$$L_{pq}(f) = \sum_{m=-\infty}^{\infty} \frac{|G(f - m/T)|^2\Phi_p^*(f - m/T)\Phi_q(f - m/T)}{K_{nn}(f - m/T)}. \tag{3-23b}$$

In the examples to follow, we shall restrict consideration to problems in which the noise is white (power spectral density N_0), and there is no dispersion, i.e., $G(f) = 1$. For this special class of problems, the preceding solution reduces to the simpler form

$$\begin{aligned} h(t,s) &= \sum_{p,q} \sum_{n,m} h_{pq}(n - m)\phi_p(t - nT)\phi_q^*(s - mT) \\ H(f) &= R(f)[B(f)R(f) + N_0 I]^{-1} \\ \langle J_0 \rangle &= N_0 \int_{-1/2T}^{1/2T} \operatorname{tr}[B(f)H(f)]\, df \\ &= N_0 \operatorname{tr} h(0), \qquad \text{iff } B(f) = I. \end{aligned} \tag{3-27}$$

Note the parallel to the formulas (from Theorem 1) for the optimum time-invariant filter

$$\begin{aligned} \tilde{H}(f) &= \frac{(1/T)\Phi'(f)R(f)\Phi^*(f)}{(1/T)\Phi'(f)R(f)\Phi^*(f) + N_0} \\ \langle J \rangle_0 &= N_0 \int_{-\infty}^{\infty} \tilde{H}(f)\, df = N_0\tilde{h}(0). \end{aligned} \tag{3-28}$$

Our first two examples were presented in our preliminary paper [18]. Therefore, we shall only summarize them here. The remaining three examples are discussed in more detail.

1) *FDM Signals:* As discussed in [18], an example of a random signal that is conveniently represented by a finite-order HSR is the FDM signal composed of the sum of M separate uniformly-spaced (in frequency) phase-locked sinusoidal carriers that are each amplitude modulated with individual stationary signal processes (power spectral densities $\{K_p\}$). If the individual baseband signals are uncorrelated, then the general structure of the optimum filter provided by Theorem 3 reduces to M parallel paths. The pth path is composed of a sine wave demodulator at the input, a Wiener filter for the pth baseband signal, and a sine wave modulator at the output. Thus the optimum time-varying filter demultiplexes and demodulates the FDM signal, optimally filters the baseband components, and then modulates and multiplexes the filtered components. Using the results of Theorems 1 and 3, it was shown that as the signal-to-noise ratio (SNR) increases, the improvement factor P (3-5) approaches 2. It was also shown that because of the similarity in the formulas for $\langle J_0 \rangle$ and $\langle J \rangle_0$, the two filters can be compared on a channel-by-channel basis and large differences among the statistics of the baseband signals will not affect the comparison. This is not the case in the TDM scheme discussed in the following example.

2) *TDM Signals:* As discussed in [18], an example of a random signal that is conveniently represented by a finite-order TSR is the TDM signal composed of the sum of M separate PAM signals that are uniformly interleaved with a time separation of T/M. If the individual baseband signals (with power spectral densities—before sampling—denoted by $\{K_p\}$) are uncorrelated, and the pulse translates are orthogonal, then the general structure of the optimum filter provided by Theorem 2 (or (3-27)) reduces to M parallel paths. The pth path is composed of a matched filter and periodic sampler, a discrete Wiener filter for the pth amplitude sequence, and a pulse-generating filter at the output. Thus the optimum time-varying filter (paralleling that for the FDM signal) demultiplexes and demodulates the TDM signal, optimally filters the sample sequences, and then modulates and multiplexes the filtered sequences. Using the results of Theorems 1 and 2 (or (3-27) and (3-28)), it was shown that the difference between $\langle J_0 \rangle$ and $\langle J \rangle_0$ narrows down to a term in the denominator of the integrand. One conclusion based on this simple difference is that if all the processes multiplexed are statistically identical, then the TDM signal is WSS, and there is no improvement ($P = 1$). On the other hand, if the individual processes are very different, then the degree of CS can be large, and there

can be substantial improvement. For example, if the M processes have disjoint power spectral densities, then there is a low-noise improvement of $P = M$. Similarly, if all but one of the M baseband signals should happen to vanish, then there is a low-noise improvement factor of $P = M$.

3) *PAM Signals:* In this example, we use the PAM signal (1-2), which admits a first-order ($M = 1$) TSR, to illustrate several important characteristics of CS processes. To begin with, we consider full-duty-cycle unit-energy rectangular pulses. This PAM signal has constant mean and variance. Assuming that the power spectral density $K(f)$ of the WSS process—with sample values that amplitude-modulate the pulses—is bandlimited to $[-1/2T, 1/2T]$, we obtain from (3-27) the following time-averaged value of the minimum filtering error

$$\langle J_0 \rangle = \int_{-1/2T}^{1/2T} \frac{N_0 K(f)}{N_0 + K(f)}\, df \tag{3-29}$$

and we obtain from (3-28) the following minimum value of the time-averaged filtering error

$$\langle J \rangle_0 = \sum_{n=-\infty}^{\infty} \int_{-1/2T}^{1/2T} \frac{N_0 K(f)[\sin(\pi(Tf-n))/\pi(Tf-n)]^2}{N_0 + K(f)[\sin(\pi(Tf-n))/\pi(Tf-n)]^2}\, df. \tag{3-30}$$

Now, for high SNR's, $\langle J_0 \rangle \simeq N_0/T$, and for any number I there exists an N_0 small enough to guarantee that $\langle J \rangle_0 > IN_0/T$, so that $P > I$. Thus we see that although the signal has constant mean and variance, the optimum time-varying filter significantly outperforms the optimum time-invariant filter. For example, an improvement factor of $P > I = 10$ results with an SNR of $\sigma^2 T/N_0 = 10^4$, where we have chosen $K(f) = \sigma^2 T$, for all $f \in [-1/2, 1/2T]$. We summarize this result by saying that the degree of CS of a signal with constant mean and variance need not be small.

We now consider zero-mean PAM with unit-energy sinc pulses of bandwidth I/T. From (3-27), (3-28), assuming constant power spectral density bandlimited to $[-1/2T, 1/2T]$, we obtain

$$\langle J_0 \rangle = \frac{\sigma_s^2 \sigma_n^2 / I}{\sigma_s^2 + \sigma_n^2 / I}$$

$$\langle J \rangle_0 = \frac{\sigma_s^2 \sigma_n^2}{\sigma_s^2 + \sigma_n^2} \tag{3-31}$$

where σ_s^2 is the time-averaged signal variance and $\sigma_n^2 = N_0 I/T$ is the noise variance. Notice that the effective noise variance is reduced by the factor $1/I$ for the time-varying filter. If we fix the SNR at unity, then the improvement factor is $P = (I + 1)/2$. Thus we see that there is a substantial improvement in performance at this low SNR, provided that I is much greater than one. ($I = 1$ corresponds to transmission at the Nyquist rate and renders the PAM signal WSS.) We conclude that high SNR is not a prerequisite for improved performance with time-varying filters.

Next, we discuss PAM with less-than-full-duty-cycle rectangular unit-energy pulses of width T/I. Assuming uncorrelated pulse amplitudes, we obtain from (3-27), (3-28) the following estimation errors:

$$\langle J_0 \rangle = \frac{\sigma_s^2 N_0/T}{\sigma_s^2 + N_0/T}$$

$$\langle J \rangle_0 = \int_{-\infty}^{\infty} \frac{N_0 \sigma_s^2 |\Phi(f)|^2}{N_0 + \sigma_s^2 |\Phi(f)|^2}\, df > \frac{\sigma_s^2 I N_0/T}{\sigma_s^2 + I N_0/T}. \tag{3-32}$$

For high SNR's, there is an improvement factor $P > I$ that increases as the pulsewidth decreases. Thus improvement increases with increasing degree of CS.

More detailed results on the solution (3-27) for the optimum time-varying filters for PAM signals are given in [21].

4) *Synchronous M-ary Data Signals:* An important class of signals that is of practical interest in digital communications is the class of synchronous M-ary signals defined in (1-3). In this expression, $\{a_n\}$ (hereafter denoted $\{b_n\}$) is an M-ary random sequence, each random variable of which has the alphabet of realizations $\{\alpha_1,\alpha_2,\cdots,\alpha_M\}$. This class of signals includes FSK and PSK signals, and PPM, PWM, and AM signals. These synchronous M-ary signals admit Mth order TSR's where the pth basis function is $\phi_p(t) = \phi(t,\alpha_p)$ and where the M jointly WSS sequences $\{a_p(n)\}$ comprise indicator sequences. That is, for every integer n, the realizations of all but one of the M elements $\{a_1(n), a_2(n),\cdots,a_M(n)\}$ are zero, and the nonzero realization is equal to one. For this TSR, the matrix of correlation sequences, with elements $\{r_{pq}(n - m)\}$, is a matrix of joint probabilities of which the $(n - m)$th element of the pqth sequence is the joint probability of the event $b_n = \alpha_p$ and $b_m = \alpha_q$. Also, the mean value of the random variable $a_p(n)$ is just the probability of the event $b_n = \alpha_p$.

As a specific example of a synchronous M-ary data signal we consider the FSK signal

$$x(t) = \sum_{n=-\infty}^{\infty} w(t - nT) \cos(2\pi b_n f_0 t) \tag{3-33}$$

where w is a unit-energy sinc pulse of bandwidth $B \le f_0 =$ integer multiple of $1/T$, and where the frequency parameters $\{b_n\}$ are statistically independent M-ary random variables with equiprobable realizations $\{1,2,\cdots,M\}$. The Mth order TSR for the centered version $(x - E\{x\})$ of this process has a constant spectral density matrix with elements

$$R_{pq}(f) = \frac{\delta_{pq}}{M} - \frac{1}{M^2} \tag{3-34}$$

where δ_{pq} is the Kronecker delta.

Now let us consider the problem of optimally filtering this FSK signal out of white noise. Employing (3-34) in (3-27) and (3-28) results in the estimation error variances

$$\langle J_0 \rangle = \frac{1}{T} \frac{N_0(M-1)}{MN_0 + 1}$$

$$\langle J \rangle_0 = \frac{1}{T} \frac{N_0(M-1)}{MN_0 + (M-1)/2BTM}. \tag{3-35}$$

Using an effective noise bandwidth of $2BM$ to obtain a noise variance of $2BMN_0$ and using the time-averaged signal variance $(M-1)/MT$, we express the improvement factor in terms of the SNR ρ

$$P = \frac{1+\rho 2BTM/(M-1)}{1+\rho}. \tag{3-36}$$

The lowest practical bandwidth $B = 1/T$ corresponds to transmitting pulses at the Nyquist rate, and results in a minimum improvement factor of 4 for binary FSK at high SNR's. As the bandwidth of the pulse is increased in order to decrease intersymbol interference, the high-ρ improvement factor increases in direct proportion. Hence the time-varying filter becomes more attractive, relative to the time-invariant filter, as the quality of transmission increases. Note that for SNR's as low as unity, there can still be a substantial improvement.

Detailed results on the solutions (3-27) for optimum time-varying filters for synchronous M-ary data signals are given in [21].

5) *Video Signals:* For this example, we consider the process that results from scanning a two-dimensional visual pattern using the conventional rectangular scanning format (without interlace). The visual pattern is modeled by a two-dimensional random step function, giving a stationary autocorrelation with exponential form that is separable in the horizontal and vertical directions [22]. Neglecting frame-to-frame correlation, the scanner output is a CS (T) process, where T is equal to the line-scan interval. Consider any two time instants t_1 and t_2, where t_2 occurs in the mth line after the one that contains t_1. Then the scanner output has a normalized autocorrelation function given by

$$k_{xx}(t_1,t_2) = \alpha^m \exp(-2\pi f_0|t_1 - t_2 + mT|) \tag{3-37}$$

where the parameter f_0 characterizes correlation in the horizontal direction and α is the line-to-line correlation (m is a function of t_1,t_2).

The Karhunen–Loève TSR has proven to be especially useful for representing this process. For this representation, the $\phi_p(t)$ are the normalized solutions of

$$\int_0^T \exp(-2\pi f_0|t-s|)\phi_p(s)\,ds = \lambda_p\phi_p(t), \quad \text{for all } t \in [0,T].$$

The eigenfunctions of this equation are cosine and sine functions with frequencies that are given by the solutions of

$$\tan(\pi T f_p) = \frac{f_0}{f_p} \qquad \tan(\pi T f_p) = \frac{-f_p}{f_0},$$

respectively, and the corresponding eigenvalues are given by [1]

$$\lambda_p = \frac{1}{\pi f_0}\left[1 + \left(\frac{f_p}{f_0}\right)^2\right]^{-1}. \tag{3-38}$$

The matrix of crosscorrelations for this TSR has the particularly simple form

$$r(n) = \alpha^{|n|}\Lambda \tag{3-39}$$

TABLE I
PERFORMANCE OF OPTIMUM TIME-VARYING FILTER FOR VIDEO SIGNAL: $10\log_{10}(\rho\langle J_0\rangle)$ dB, FOR TYPICAL VALUES OF LINE-TO-LINE CORRELATION AND SNR ρ

α \ ρ	10	10^2	10^3	10^4
.98	9.7	6.6	3.3	1.0
.95	7.3	3.9	1.4	0.2
.9	5.2	2.2	0.5	~
.8	3.3	0.9	0.1	~
.7	2.3	0.5	~	~

where $\Lambda = \operatorname{diag}\{\lambda_p\}$. Thus the spectral density matrix is

$$R(f) = \left[\frac{1-\alpha^2}{(1-\alpha)^2 + 4\alpha\sin^2(\pi T f)}\right]\Lambda. \tag{3-40}$$

Now consider the problem of continuous waveform estimation when this signal is received with additive white noise. For this case, (3-27) is expressed as

$$\langle J_0\rangle = \frac{N_0}{T}\sum_{p=1}^{\infty}\lambda_p\left|\left[\lambda_p + N_0\left(\frac{1+\alpha}{1-\alpha}\right)\right]\cdot\left[\lambda_p + N_0\left(\frac{1-\alpha}{1+\alpha}\right)\right]\right|^{-1/2}. \tag{3-41}$$

This performance has been numerically evaluated for the specific case of a 500-line square format with equal correlation in the visual pattern along horizontal and vertical directions. This requires that $\alpha = \exp(-2\pi f_0 T/500)$. Assuming approximately equal resolution requirements in the horizontal and vertical directions, the bandwidth of the signal is approximately $500/2T$. Using this bandwidth, we approximate (3-41) by the contribution of only the first 500 terms, and we take $\rho = T/500N_0$ as a measure of the signal-to-noise power ratio. In Table I, the performance of the optimum filter for typical values of α and ρ is indicated by $-10\log_{10}(\rho\langle J_0\rangle)$. Since we have assumed a unit-variance signal, this quantity represents the improvement (in decibels) in optimum filtering over that of an ideal low-pass filter that simply rejects the noise at frequencies above $500/2T$. For the parameter values chosen, the cyclic fluctuations in the correlation of the video signal are insignificant (low degree of CS), and the improvement factor P is negligible. This fact has been verified numerically by evaluation of $\langle J\rangle_0$ in (3-28) for this case.

Observe that even though the TSR for the video signal is infinite dimensional ($M = \infty$), we were still able to employ it in Theorem 2 to solve the optimum filtering problem. The reason for this is that the infinite-dimensional TSR spectral density matrix R is diagonal (3-40). This is in fact the case for various signal formats including PAM signals, TDM (not necessarily PAM) signals, and video signals. The property in common here is the Markov-like form of the autocorrelation function

$$k_{xx}(t + mT, s) = \alpha_m k_{xx}(t,s), \qquad \text{for all } t, s \in [0,T] \quad (3\text{-}42)$$

for all integers m. This form results in a spectral density matrix for the Karhunen–Loève TSR that has the diagonal form

$$R(f) = \Lambda \sum_m \alpha_m \exp(-j2\pi mfT). \quad (3\text{-}43)$$

Another form for infinite-dimensional TSR spectral density matrices that can permit matrix inversion is the sum of a diagonal matrix and the outer product of finite-rank matrices

$$R(f) = \Lambda(f) + P(f)Q'(f) \quad (3\text{-}44)$$

where Λ is diagonal and P,Q are $\infty \times M$ dimensional (M finite). It can be shown [13] that any TSR spectral density matrix for a random process which is the output of a periodically (T) time-varying Nth order linear dynamical system driven by jointly WSS white noise processes, exhibits the preceding form with $M = 2N$, Q constant, and $\Lambda = r(0)$, where $r(0)$ is diagonal if the TSR employs an orthogonal expansion on $[0,T]$.

IV. Summary

It is evident that many physical processes are more accurately modeled by a CS process than by a stationary process. Fortunately, the CS process has properties that make it a great deal less formidable in analysis than the general nonstationary process. In fact, techniques used for wide-sense stationary processes are generally applicable since scalar-valued CS processes can be represented by vector-valued jointly WSS processes. Furthermore, the representors may be either of the continuous or discrete type. Some of the representations discussed here when applied to generally nonstationary processes affect a decomposition into representors that, although not WSS, are still more elementary than the process being represented.

The value of using the more accurate CS models in place of time-averaged (or phase-randomized) WSS models for processes is demonstrated with examples that illustrate optimum-filter performance improvement. The CS models have also proven valuable in the design and analysis of optimum filtering structures and of synchronization schemes employed to extract timing information from received CS signals.

The practical utility of the representations for CS processes that we discuss is illustrated in three ways. First, the representations give insight into structural properties of CS processes and exploit their similarities to WSS processes. Second, our examples show that the representations provide natural models for many communication signal formats. Third, they lead to analytical solutions and result in practical interpretations of optimum periodically time-varying filters for CS processes.

A basic property of a CS process that is of practical interest is its degree of CS. This property might be interpreted as the "distance" to the "nearest" stationary process. This distance measure could be formulated in a variety of ways, depending on the application at hand. Our examples show that such simple measures as the amount of fluctuation of the mean and the variance over one cycle may not be adequate to characterize the degree of CS. A possible measure is the relative performances of the optimum filters for a CS process and its phase-randomized version in the case of additive white noise. This measure provides useful information for designing scanning or multiplexing formats and for deciding whether or not to employ time-variable signal processing operations. Its drawbacks are that it is often not a convenient measure to evaluate numerically and that it is dependent on the level of the noise interference.

Further work is needed on representations of processes that are not strictly CS but could be modeled similarly using interval and phase parameters that drift slowly with time.

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