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The Structure of Least-Mean-Square Linear Estimators for Synchronous M-ary Signals

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***Abstract*—Several authors have shown that the structure of the least-mean-square linear estimator of the sequence of random amplitudes in a synchronous pulse-amplitude-modulated signal that suffers intersymbol**

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interference and additive noise is a matched filter whose output is periodically sampled and passed through a transversal filter (tapped delay line). It is our purpose in this paper to generalize this result to synchronous *m*-ary signals (e.g., FSK, PSK, PPM signals). We prove that the structure of the least-mean-square linear estimator of the sequence of random parameters in a synchronous *m*-ary signal, which suffers intersymbol interference and additive noise, is a parallel connection of *m* matched filters followed by tapped delay lines. A similar structure is derived for the continuous waveform estimator of a synchronous *m*-ary signal. Finally, we present a structure for estimation-decision detection of synchronous *m*-ary signals, which is based on least-mean-square linear estimates of *a posteriori* probabilities.

I. Introduction

We are primarily concerned here with least-mean-square (LMS) linear estimation of cyclostationary synchronous m-ary signals that have been transmitted through a noisy dispersive channel composed of a linear deterministic time-invariant transformation and additive uncorrelated wide-sense-stationary (WSS) noise. With this model for the channel, the received signal y is given by the expression

$$y(t) = \int_{-\infty}^{\infty} g(t-\sigma)s(\sigma)\,d\sigma + u(t), \tag{1}$$

where g is the impulse response for the channel transformation [transfer function $G(f)$], u is the additive noise [power spectral density $K_{uu}(f)$], and s is the transmitted signal.

We will consider only the class of signals that take the form

$$s(t) = \sum_{n=-\infty}^{\infty} p(t-nT,\, a_n), \tag{2}$$

where $\{a_n\}$ is a random sequence, the nth element of which "modulates" the pulse $p(t-nT,\cdot)$. The random variables comprising $\{a_n\}$ are discrete with m-allowable levels $\{\alpha_j; j = 1,2,\cdots, m\}$, and the random sequence is stationary of order 2. That is, the probability that $a_n = \alpha_j$ is $p(j)$ for all integers n, and the probability that $a_n = \alpha_i$ and $a_q = \alpha_j$ jointly is $p_{n-q}(i,j)$, for all integers n and q. This stationary model for $\{a_n\}$ guarantees that the random signal s is cyclostationary with period T [9]. Equation (2) for s is an idealized model for many synchronous pulse-train formats such as frequency-shift-keying (FSK), phase-shift-keying (PSK), pulse-width-modulation (PWM), pulse-position-modulation (PPM), and others.

If the modulation of $p(t-nT,\cdot)$ with a_n is linear, $p(t-nT, a_n) \equiv a_n p(t-nT)$), then s is a synchronous m-ary pulse-amplitude-modulated (PAM) signal. For this special case several authors (most notably Berger, Tufts, and Kaye and George [1]–[3]) have shown that the structure of the LMS linear estimator of the random sequence of amplitudes $\{a_n\}$ is a matched filter whose output is periodically sampled (every T seconds) and passed through a transversal filter [tapped delay line, (TDL)]. The transfer function for the matched filter is

$$M(f) = \frac{G^*(f)P^*(f)}{K_{uu}(f)} \tag{3}$$

and the transfer function for the TDL is

$$T(f) = \frac{\alpha' Q(f)\alpha}{1 + \alpha' Q(f)\alpha L(f)} = \sum_{i=-\infty}^{\infty} c_i \exp(j2\pi iTf), \tag{4}$$

where the elements of the $m \times m$ matrix Q are

$$Q_{iq}(f) = \sum_{n=-\infty}^{\infty} p_n(i,q)\exp(j2\pi nTf), \tag{5}$$

and α' is the transpose of the m-vector with elements $\{\alpha_i\}$, and

$$L(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} K_{uu}(f-n/T)|M(f-n/T)|^2. \tag{6}$$

The Fourier coefficients $\{c_i\}$ are the tap weights.

Note that the tap weights are difficult to solve for, and even if they were known, the estimator is noncausal. The real value of this solution is the identification of the structure of the optimum linear estimator. Knowledge of this structure provides a basis for developing receivers that can be near optimum and practical for implementation (e.g., a causal approximation to the matched filter M followed by the LMS N-tap TDL and an m-level threshold detector). Furthermore—and perhaps most important—knowledge of the structure provides a basis for developing adaptive receivers [4]–[7].

With this as motivation, we proceed to solve for the LMS linear estimator for the general case of m-ary nonlinear modulation.

II. Periodic Parameter Estimation

It is well known that the necessary and sufficient condition on the impulse-response function for the LMS linear estimator for the modulating sequence $\{a_n\}$ can easily be derived from the projection theorem [8], and can be expressed as the linear integral equation

$$\int_{-\infty}^{\infty} h(nT,\sigma)k_{yy}(\sigma,t)\,d\sigma = k_{ya}(t,nT), \qquad \begin{cases}\forall t \in (-\infty,\infty)\\ \forall \text{ integer } n,\end{cases} \tag{7}$$

where k_{yy} is the autocorrelation function for the received signal y

$$k_{yy}(\sigma,t) = \sum_{i=-\infty}^{\infty}\sum_{j=1}^{m} w(t-iT,\alpha_j)w_j(\sigma-iT) + k_{uu}(\sigma-t), \tag{8}$$

and k_{ya} is the cross-correlation function for y and $\{a_n\}$

$$k_{ya}(t,nT) = \sum_{i=-\infty}^{\infty}\sum_{j=1}^{m} b^j_{i-n} w(t-iT,\alpha_j). \tag{9}$$

In the preceding expressions, $\{w_j\}$ are the inverse Fourier transforms of the elements of the m-vector

$$\boldsymbol{W}(f) = \boldsymbol{Q}(f)\boldsymbol{W}(f,\alpha) = \boldsymbol{Q}(f)\boldsymbol{P}(f,\alpha)G(f), \tag{10}$$

where the elements of $\boldsymbol{P}(f,\alpha)$ are the Fourier transforms of the pulses $\{p(t,\alpha_j)\}$ and the elements of $\boldsymbol{W}(f,\alpha)$ are the Fourier transforms of the elements $\{w(t,\alpha_j)\}$, and where

$$b_n^j = \sum_{q=1}^{m} Q_n^{jq}\alpha_q. \tag{11}$$

In the preceding integral equation, h is the impulse-response function for the LMS estimator whose outputs at the times $\{nT\}$ are the estimates $\{\hat{a}_n\}$ of the parameters $\{a_n\}$.

Now, it is shown in the Appendix that the solution to (7) is time-invariant and has Fourier transform—the transfer function for the estimator—given by the formula

$$H(f) = \sum_{q=1}^{m} M_q(f)T_q(f), \tag{12}$$

where

$$M_q(f) = \frac{G^*(f)P^*(f,\alpha_q)}{K_{uu}(f)} \tag{13}$$

and the m-vector with elements $T_q(f)$ is

$$\boldsymbol{T}(f) = [\boldsymbol{I} + \boldsymbol{Q}(f)L(f)]^{-1}\boldsymbol{Q}(f)\alpha = \sum_{i=-\infty}^{\infty} c_i \exp(j2\pi iTf), \tag{14}$$

where I is the $m \times m$ identity matrix and L is given by

$$L(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} K_{uu}(f - n/T)M^*(f - n/T)M'(f - n/T). \quad (15)$$

Clearly, M_q is the matched filter for the dispersed version of the pulse $p(t,\alpha_q)$ in additive colored noise. Also, since $T(f)$ is $1/T$-periodic then $T_q(f)$ is a TDL with tap weights $\{c_i^q\}$. Hence, the structure of periodic parameter estimators for synchronous m-ary signals is the parallel connection of m matched filters each in tandem with a periodic sampler and a TDL as shown in Fig. 1. This structure directly parallels that for PAM. In fact, the formula for the m-vector of TDL's is simply a vector-version of the scalar formula for PAM (4), and this general solution when applied to the special case of PAM reduces to the well-known formula of (4).

In general, solving for the tap weights $\{\mathbf{c}_i\}$ will be even more difficult (due to the matrix inverse) than it is for PAM. Also, the estimator is noncausal. Again, as for the case of PAM, the real value of our solution is the identification of the structure of the optimum estimator. Knowledge of this optimum structure provides a basis for developing adaptive receivers and fixed receivers that can be near-optimum and practical for implementation. For example, if we impose the constraint that our estimator consist of a parallel connection of causal approximations to the m matched filters followed by N-tap transversal filters, then the mean-squared error is a quadratic functional on R^{mN}. Thus, the LMS set of tap weights can be found by inverting an $mN \times mN$ constant matrix.

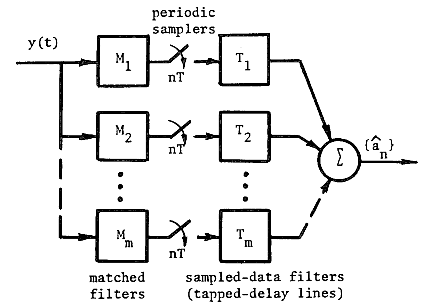

Fig. 1. Structure for LMS periodic parameter estimation on synchronous m-ary signals.

III. Continuous Waveform Estimation

If it is desired to estimate the total synchronous m-ary waveform rather than the sequence of parameters then we must solve a new integral equation for the LMS linear estimator [which will be a periodically (T) time-varying system]. The optimum continuous waveform estimator for PAM can be obtained by simply generating pulses with amplitudes equal to the estimates obtained from the periodic parameter estimator. Similarly, it will be shown that the optimum continuous waveform estimator for nonlinearly modulated synchronous m-ary signals can be obtained by modifying the internal structure of the periodic parameter estimator to include m pulse generators.

Using the projection theorem, we obtain the following integral equation for the optimum linear estimator for s

$$\int_{-\infty}^{\infty} h(\tau,\sigma)k_{yy}(\sigma,t)\, d\sigma = k_{ys}(t,\tau), \qquad \forall t,\tau \in (-\infty,\infty), \quad (16)$$

where k_{ys} is the cross-correlation function for y,s

$$k_{ys}(t,\tau) = \sum_{i=-\infty}^{\infty} \sum_{q=1}^{m} p(\tau - iT, \alpha_q)w_q(t - iT). \quad (17)$$

Now a method of solution [9], which parallels that given in the Appendix, results in the following explicit solution for the optimum impulse response function

$$h(t,\tau) = \sum_{i=-\infty}^{\infty} \sum_{q=1}^{m} p(t - iT, \alpha_q)h_q(iT - \tau), \quad (18)$$

where the Fourier transforms of the functions $\{h_q\}$ are given by the formulas

$$H_q(f) = \sum_{p=1}^{m} M_p(f)T_{pq}(f), \quad (19)$$

where $\{M_p\}$ are the m matched-filter transfer functions of (13) and $\{T_{pq}\}$ are the transfer functions for the $m \times m$ matrix $[T(f)]$ of TDL's

$$[T(f)] = [I + Q(f)L(f)]^{-1}Q(f). \quad (20)$$

Thus, we see that the structure of the LMS continuous waveform estimator is a bank of m matched filters followed by a matrix of TDL's and an output bank of m pulse generators as shown in Fig. 2.

If we postmultiply the matrix $[T(f)]$ of (20) by the vector α, then we obtain the vector $T(f)$ of TDL's, which is given in (14) and is employed in the periodic parameter estimator derived in the previous section

$$T(f) = [T(f)]\alpha. \quad (21)$$

Hence, if the m modulators at the output of the waveform estimator (Fig. 2) are replaced by m attenuators with attenuations $\{\alpha_q\}$, then the resultant time-invariant filter will have an impulse response that is identical to that of the periodic parameter estimator of (12) (Fig. 1).

IV. Detection

From the preceding discussion we see that the estimates provided by both the parameter estimator and the waveform estimator can be expressed as weighted sums

$$\hat{a}_n = \sum_{q=1}^{m} b_q(nT)\alpha_q \quad (22)$$

$$\hat{s}(t) = \sum_{n=-\infty}^{\infty} \sum_{q=1}^{m} b_q(nT)p(t - nT, \alpha_q), \quad (23)$$

where $\{b_q(nT); q = 1,2,\cdots,m\}$ are the m outputs provided every T seconds by the matrix of TDL's in Fig. 2. So we see that these two types of estimators both employ the same component, denoted H_o in Fig. 2 to compute the m sequences of "weights" $\{b_q(nT)\}$, and differ only in that the sequence estimator employs the weights to compute an "average alphabet-letter" every T seconds, and the waveform estimator employs the weights to compute an "average pulse" every T seconds.

In fact, it can be shown that the $\{b_q(nT)\}$ are LMS linear estimates of the *a posteriori* probabilities of reception $\{\Pr[a_n = \alpha_q/y]\}$; i.e., $b_q(nT)$ is the LMS estimate of the probability that the nth symbol transmitted was the qth letter of the alphabet, given the received signal. Hence, the estimator component H_o consisting of the matched filters and TDL's in Fig. 2 minimizes the mean-squared error $E\{[b_q(nT) - \Pr[a_n = \alpha_q/y]]^2\}$ subject to the constraint of linearity. (The expectation is over the ensemble of received signals.)

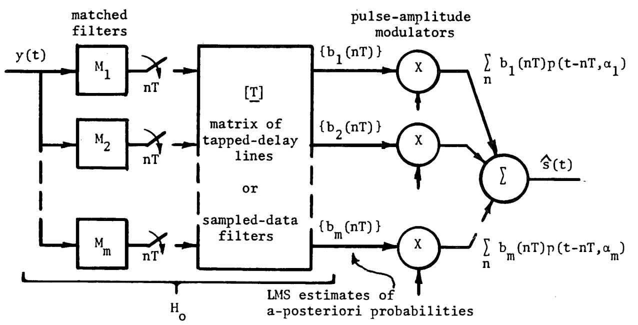

Fig. 2. Structure for LMS continuous waveform estimation on synchronous m-ary signals.

This result suggests a scheme for detection: eliminate the output pulse-amplitude modulators in Fig. 2 and feed the m probability estimates into an m-ary decision device, which treats the probability estimates as if they were the actual probabilities. Such estimation-decision detectors provide the basis for various adaptive detection schemes. Note that the LMS error criterion for estimating the probabilities is particularly appropriate when the estimates are used to make decisions, since large errors—which are likely to result in wrong decisions—are weighted more heavily by this criterion than small errors—which are likely to lead to correct decisions.

Appendix

We assume a form for the impulse-response function, which is similar to that of the correlation functions

$$h(kT,\sigma) = \sum_{i=-\infty}^{\infty} \sum_{p=1}^{m} c_i{}^p \theta_p(kT - (\sigma - iT)). \tag{A-1}$$

We see that the estimator is time-invariant since h depends only on the difference of its arguments. The Fourier transform of h (the transfer function) is

$$H(f) = \sum_{p=1}^{m} M_p(f) T_p(f),$$

where M_p is the Fourier transform of $\theta_p(t)$ and

$$T_p(f) = \sum_i c_i^p \exp(j2\pi iTf). \tag{A-2}$$

With some foresight, we choose

$$M_p(f) = \frac{G^*(f)P^*(f,\alpha_p)}{K_{uu}(f)}. \tag{A-3}$$

Now, changing the index on the infinite sum in (A-1) results in the alternate expression

$$h(kT,\sigma) = \sum_{i=-\infty}^{\infty} \sum_{p=1}^{m} c_{i-k}^p \theta_p(iT - \sigma),$$

which, if substituted along with (8) and (9) into (7) yields

$$\sum_{i=-\infty}^{\infty} \sum_{q=1}^{m} w(t - iT, \alpha_q) \left[\sum_{j=-\infty}^{\infty} \sum_{p=1}^{m} c_{j-k}^p \cdot \int_{-\infty}^{\infty} w_q(\sigma - iT)\theta_p(jT - \sigma)\, d\sigma + c_{i-k}^q - b_{i-k}^q \right] = 0, \qquad \forall t,k.$$

This equation will be satisfied if the expression in brackets is zero for all i,k,q

$$\sum_{j=-\infty}^{\infty} \sum_{p=1}^{m} c_{j-k}^p D_{i-j}^{qp} + c_{i-k}^q - b_{i-k}^q = 0, \qquad \forall i,k,q, \tag{A-4}$$

where

$$D_r{}^{qp} \triangleq \int_{-\infty}^{\infty} w_q(-rT - \sigma)\theta_p(\sigma)\, d\sigma. \tag{A-5}$$

Letting $j - k = n$, $i - k = r$ in (A-4) yields

$$\sum_{p=1}^{m} \sum_{n=-\infty}^{\infty} D_{r-n}^{qp} c_n{}^p + c_r{}^q - b_r{}^q = 0, \qquad \forall r,q. \tag{A-6}$$

We denote the bilateral z transform of the sequence (indexed by n) $\{c_n{}^p\}$ as

$$\tilde{c}_p(z) \triangleq \sum_{n=-\infty}^{\infty} c_n^p z^{-n}.$$

Now, taking the z transform of both sides of (A-6) yields

$$\sum_{p=1}^{m} \tilde{D}_{qp}(z)\tilde{c}_p(z) + \tilde{c}_q(z) = \tilde{b}_q(z)$$

or in matrix notation

$$[\tilde{\boldsymbol{D}}(z) + \boldsymbol{I}]\tilde{\boldsymbol{c}}(z) = \tilde{\boldsymbol{b}}(z).$$

It can be shown that $\tilde{\boldsymbol{D}}$ is nonnegative definite, so the indicated matrix inverse exists and we have

$$\tilde{\boldsymbol{c}}(z) = [\tilde{\boldsymbol{D}}(z) + \boldsymbol{I}]^{-1}\tilde{\boldsymbol{b}}(z). \tag{A-7}$$

From (A-2), (5) and (11), we find that $\tilde{\boldsymbol{c}}(\exp[-j2\pi Tf]) = \boldsymbol{T}(f)$ $\tilde{\boldsymbol{b}}(\exp[-j2\pi Tf]) = \boldsymbol{Q}(f)\boldsymbol{\alpha}$. Thus,

$$\boldsymbol{T}(f) = \left[\sum_{r=-\infty}^{\infty} \boldsymbol{D}_r \exp(j2\pi rTf) + \boldsymbol{I} \right]^{-1} \boldsymbol{Q}(f)\boldsymbol{\alpha}. \tag{A-8}$$

But, using (A-5), it can be shown that

$$\sum_{r=-\infty}^{\infty} \boldsymbol{D}_r \exp(j2\pi rTf) = \boldsymbol{Q}(f)\boldsymbol{L}(f),$$

where $\boldsymbol{Q},\boldsymbol{L}$ are defined in (5) and (15). Hence, $\boldsymbol{T}(f)$ is given by (14).

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