"Modulation-Rate" Distortion in Frequency Modulators

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Abstract—An analysis of "modulation-rate" distortion originating in frequency modulators is presented. The analysis is based on the solutions of the differential equations representing two fundamental classes of variable frequency oscillators. It is shown that the VFOs of one class produce ideal frequency modulation, while the VFOs of the other class inherently produce distortion. It is shown that this distortion is a function of the ratio of two rates, the rate of change of energy storage capacity and the rate of change of stored energy.

Introduction

A FREQUENCY modulator can be modeled as a two-port with a modulating signal V_{mod} input, and a modulated signal V_{FM} output. Ideally, the output signal would be given by the following expression:

$$V_{\text{FM}} = A_0 \cos\left[K \int V_{\text{mod}}\, dt + \theta\right] \tag{1}$$

where A_0, K, θ are constants. The frequency $\omega(t)$ of the output signal is the derivative of the argument and, for the ideal modulator, is proportional to the modulating signal

$$\omega(t) = \frac{d}{dt}\left[K \int V_{\text{mod}}\, dt + \theta\right] = KV_{\text{mod}}. \tag{2}$$

In general, $V_{\text{mod}} \propto [1 + \delta(t)]$; therefore, the modulated frequency becomes $\omega(t) = \omega_c + \omega_c\delta(t)$ where ω_c is commonly referred to as the carrier frequency.

For nonideal frequency modulators, the output signal as shown in (1) must be modified by adding amplitude modulation and frequency distortion. That is,

$$V_{\text{FM}} = A(t) \cos\left[\int (KV_{\text{mod}} + D(t))\, dt\right] \tag{3}$$

where $A(t)$ is the amplitude modulation and $D(t)$ is the frequency distortion.

For many applications, the undesired amplitude modulation is of less concern than the frequency distortion, because under certain conditions the amplitude modulation can be removed by limiting and filtering [1]. Of prime importance is the frequency distortion, because in most cases it cannot be separated from the desired signal [1].

There are several sources of distortion in frequency modulators: nonlinearities, thermal and shot noise, method of frequency deviation, characteristics of the frequency deviating elements, etc. This paper deals with distortion resulting from one source: method of frequency deviation. It is the purpose of this paper to present an analysis of this particular distortion, to be called "modulation-rate" distortion (abbreviated to rate distortion[1]), to identify the mechanism by which this distortion is created, and finally to evaluate various methods of frequency deviation with respect to rate distortion performance.

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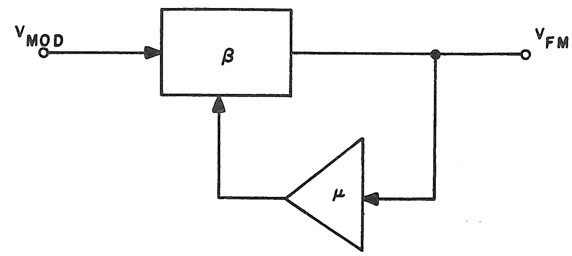


Fig. 1. Basic VFO.

Models and Differential Equations of VFOs

In essence, a frequency modulator is a variable frequency oscillator (VFO). The general form of a VFO is shown in Fig. 1, where β is a voltage-variable frequency-selective network, its transfer characteristic being controlled by the input modulating signal V_{mod}; and μ is an amplifier that compensates for losses in β such that the loop is essentially lossless.

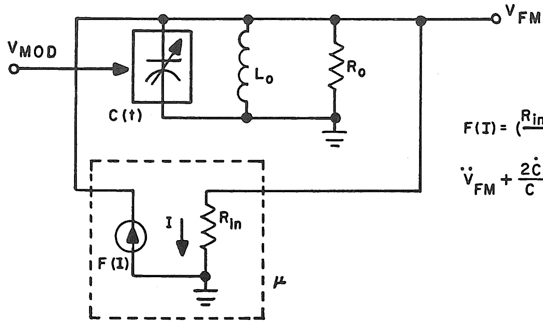
Four fundamental forms of frequency-selective networks will be considered:

1) parallel antiresonant *LC*
2) series resonant *LC*
3) *RC*-type Wien bridge
4) *RC*-type bridged T.

Ten different VFOs may be constructed from these four fundamental forms. Circuit models of these ten VFOs and the differential equations representing each are shown in Fig. 2. In these circuit models, the amplifier μ has been modeled by an ideal current or voltage amplifier. The derivation of a typical differential equation is shown in Appendix I. In the derivation of these differential equations, one idealizing assumption has been made: the gain function of μ has been assumed to be a constant such that the losses in β are exactly canceled. In other words, it has been assumed that the VFOs are oscillating in equilibrium. This assumption precludes the nonlinearity normally necessary for equilibrium.[2]

[1] The quantity hereafter referred to as "rate distortion" should not be confused with the totally unrelated quantity from information theory called "information-rate" distortion and abbreviated to rate distortion.

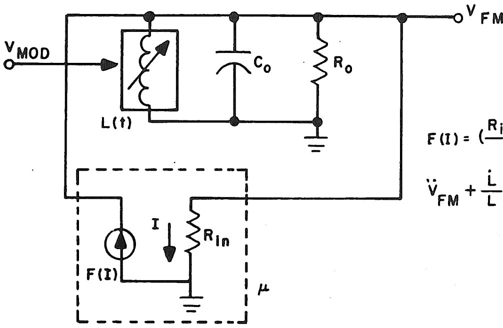
[2] This assumption has been made in order that any distortion appearing in the solution of the governing differential equation can be identified as modulation-rate distortion, that is, distortion due *only* to the method of frequency deviation.



$$F(I) = \left(\frac{R_{in} + R_o}{R_o}\right) I$$

$$\ddot{V}_{FM} + \frac{2\dot{C}}{C}\dot{V}_{FM} + \left(\frac{\ddot{C}}{C} + \frac{1}{L_oC}\right)V_{FM} = 0$$

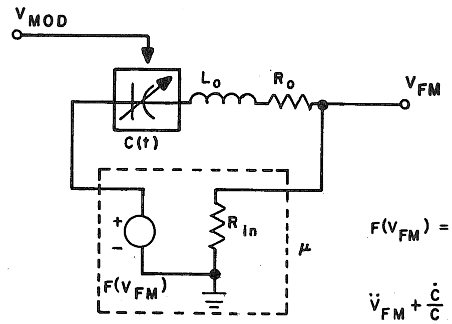
Fig. 2. (a) Parallel $LC(t)$ VFO.



$$F(I) = \left(\frac{R_{in} + R_o}{R_o}\right) I$$

$$\ddot{V}_{FM} + \frac{\dot{L}}{L}\dot{V}_{FM} + \frac{1}{LC_o}V_{FM} = 0$$

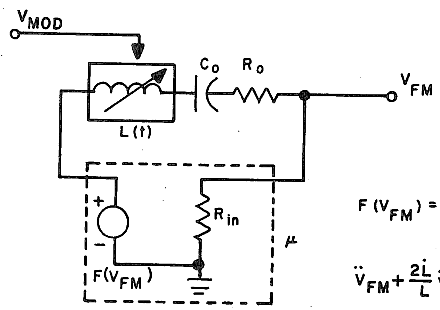
Fig. 2. (b) Parallel $L(t)C$ VFO.



$$F(V_{FM}) = \left(\frac{R_o + R_{In}}{R_{In}}\right)V_{FM}$$

$$\ddot{V}_{FM} + \frac{\dot{C}}{C}\dot{V}_{FM} + \frac{1}{L_oC}V_{FM} = 0$$

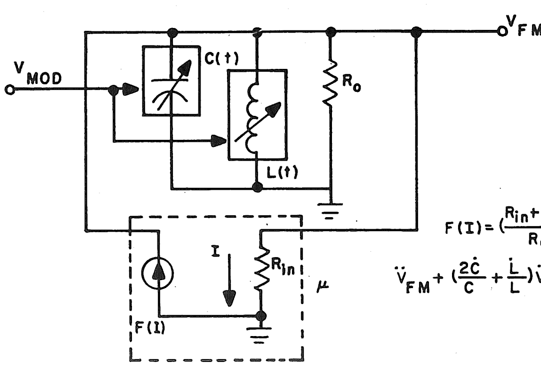
Fig. 2. (c) Series $LC(t)$ VFO.



$$F(V_{FM}) = \left(\frac{R_o + R_{In}}{R_{In}}\right)V_{FM}$$

$$\ddot{V}_{FM} + \frac{2\dot{L}}{L}\dot{V}_{FM} + \left(\frac{\ddot{L}}{L} + \frac{1}{LC_o}\right)V_{FM} = 0$$

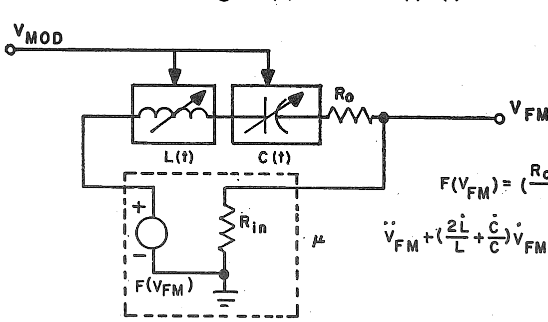
Fig. 2. (d) Series $L(t)C$ VFO.



$$F(I) = \left(\frac{R_{in} + R_o}{R_o}\right) I$$

$$\ddot{V}_{FM} + \left(\frac{2\dot{C}}{C} + \frac{\dot{L}}{L}\right)\dot{V}_{FM} + \left(\frac{\ddot{C}}{C} + \frac{\dot{L}\dot{C}}{LC} + \frac{1}{LC}\right)V_{FM} = 0$$

Fig. 2. (e) Parallel $L(t)C(t)$ VFO.



$$F(V_{FM}) = \left(\frac{R_o + R_{In}}{R_{In}}\right)V_{FM}$$

$$\ddot{V}_{FM} + \left(\frac{2\dot{L}}{L} + \frac{\dot{C}}{C}\right)\dot{V}_{FM} + \left(\frac{\ddot{L}}{L} + \frac{\dot{L}\dot{C}}{LC} + \frac{1}{LC}\right)V_{FM} = 0$$

Fig. 2. (f) Series $L(t)C(t)$ VFO.

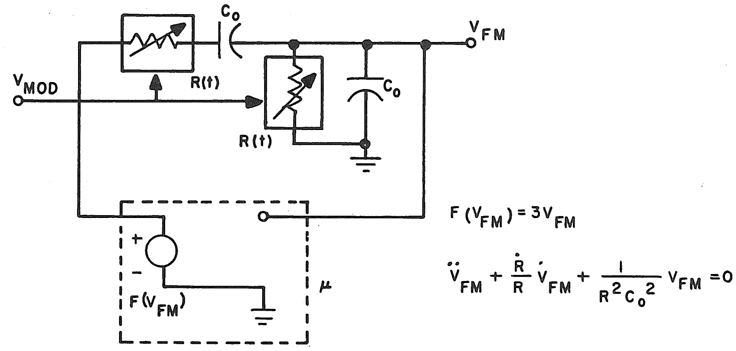


Fig. 2. (g) Wien bridge $R(t)C$ VFO.

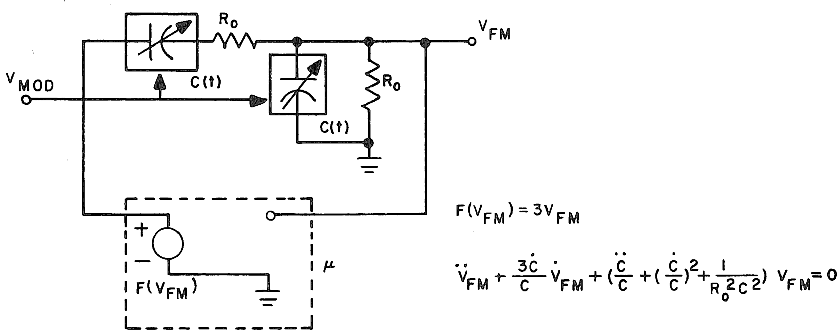


Fig. 2. (h) Wien bridge $RC(t)$ VFO.

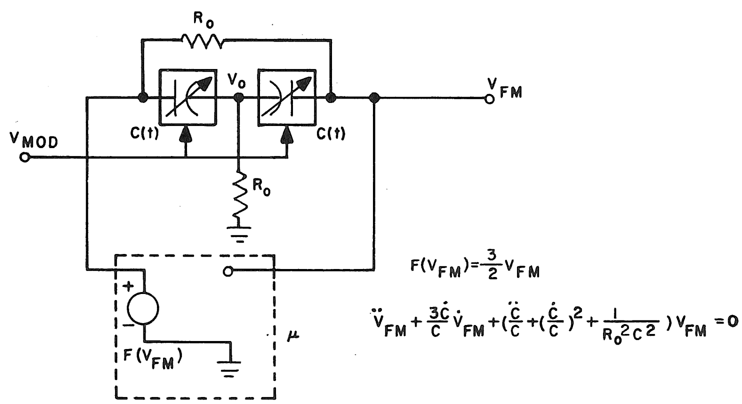


Fig. 2. (i) Bridged-T $RC(t)$ VFO.

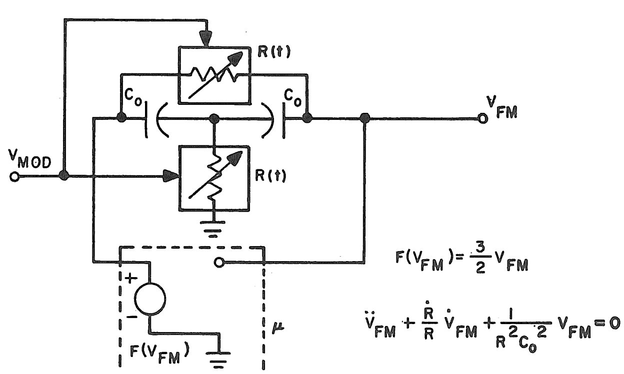


Fig. 2. (j) Bridged-T $R(t)C$ VFO.

TABLE I

Configuration	Frequency Deviating Element(s)	Change of Variables	Differential Equation
Parallel LC	$C(t)$	$y = C(t)V_{\mathrm{FM}}$, $x = (L_0C(t))^{-1}$	$\ddot{y} + xy = 0$
Parallel LC	$L(t)$	$y = L(t)V_{\mathrm{FM}}$, $x = (L(t)C_0)^{-1}$	$\ddot{y} + xy = 0$
Series LC	$L(t)$	$y = L(t)V_{\mathrm{FM}}$, $x = (L(t)C_0)^{-1}$	$\ddot{y} + xy = 0$
Series LC	$C(t)$	$y = C(t)V_{\mathrm{FM}}$, $x = (L_0C(t))^{-1}$	$\ddot{y} + xy = 0$
Wien bridge RC	$R(t)$	$y = V_{\mathrm{FM}}$, $x = (R(t)C_0)^{-1}$	$\ddot{y} - \frac{\dot{x}}{x}\dot{y} + x^2y = 0$
Wien bridge RC	$C(t)$	$y = C(t)V_{\mathrm{FM}}$, $x = (R_0C(t))^{-1}$	$\ddot{y} - \frac{\dot{x}}{x}\dot{y} + x^2y = 0$
Bridged-T RC	$C(t)$	$y = C(t)V_{\mathrm{FM}}$, $x = (R_0C(t))^{-1}$	$\ddot{y} - \frac{\dot{x}}{x}\dot{y} + x^2y = 0$
Bridged-T RC	$R(t)$	$y = V_{\mathrm{FM}}$, $x = (R(t)C_0)^{-1}$	$\ddot{y} - \frac{\dot{x}}{x}\dot{y} + x^2y = 0$
Parallel LC	$L(t) \propto C(t)$	$y = L(t)V_{\mathrm{FM}}$ or $y = C(t)V_{\mathrm{FM}}$, $x = (L(t)C(t))^{-1/2}$	$\ddot{y} - \frac{\dot{x}}{x}\dot{y} + x^2y = 0$
Series LC	$L(t) \propto C(t)$	$y = C(t)V_{\mathrm{FM}}$ or $y = L(t)V_{\mathrm{FM}}$, $x = (L(t)C(t))^{-1/2}$	$\ddot{y} - \frac{\dot{x}}{x}\dot{y} + x^2y = 0$

By making a simple change of variables in the ten differential equations shown in Fig. 2, the ten equations may be reduced to two general forms. These are shown in Table I.

Solutions of the Differential Equations

Case 1

The governing differential equation representing four of the ten VFOs is [2]

$$\ddot{y} + x(t)y = 0, \qquad \ddot{y} \triangleq \frac{d^2y}{dt^2}. \tag{4}$$

A simple means of showing that, in this case, ideal frequency modulation is not plausible is to substitute into (4) the desired solution and observe that the modulation function x, which relates the modulating signal V_{mod} to the frequency-deviating element, is far too complex to be realized with any practical element. The desired[3] solution is

$$y(t) = A(t) \cos\left[K \int V_{\mathrm{mod}}\, dt + \theta\right]. \tag{5}$$

Substituting (5) into (4) and letting $A(t) = (KV_{\mathrm{mod}})^{-1/2}$ yields the following:

$$x(t) = (KV_{\mathrm{mod}})^2 + \frac{1}{2}\left(\frac{\ddot{V}_{\mathrm{mod}}}{V_{\mathrm{mod}}}\right) - \frac{3}{4}\left(\frac{\dot{V}_{\mathrm{mod}}}{V_{\mathrm{mod}}}\right)^2$$

where x is inversely proportional to the frequency-deviating element (from Table I). Such a relationship certainly cannot be realized with any known inductive or capacitive device.

Equation (4) is Hill's equation [3] and, with the substitution $x \propto (1 + \delta(t))$, (4) becomes the Schroedinger equation [4]. In general, (4) has no closed-form solution; however, there are several possible methods of approximation [3]–[7]. Since the information sought here is an explicit expression for the modulated frequency, it would be desirable to transform (4) into another differential equation in which the transformed variable is the modulated frequency. Because (4) is a linear differential equation, the desired transformation can be accomplished by assuming a solution of the following complex form:

$$y(t) = \exp\left[i \int \omega(t)\, dt\right] \tag{6}$$

where $\omega(t)$ is complex. Substituting (6) into (4) yields the desired transformed differential equation

$$i\dot{\omega} - \omega^2 + x = 0 \tag{7}$$

where the real part of $\omega(t)$ is the modulated frequency. Equation (7) is a nonlinear differential equation. A method of approximation known as the perturbation method [6] is applicable to (7) provided x is a member of a restricted class of functions defined by convergence criteria. First, a perturbation parameter ϵ must be artificially introduced into (7). This can be accomplished by defining a new variable and modulating parameter as follows:

$$\psi \triangleq \epsilon\omega, \quad \Omega \triangleq \epsilon\sqrt{x}, \qquad (x > 0).$$

Then (7) becomes

$$\epsilon i\dot{\psi} - \psi^2 + \Omega^2 = 0. \tag{8}$$

Now it is assumed that $\psi(t)$ can be developed into a power series in ϵ, that is,

$$\psi(t) = \sum_{n=0}^{\infty} (\epsilon)^n \psi_n(t). \tag{9}$$

Substituting (9) into (8) and equating like powers of ϵ, the following sequence of differential equations is obtained.

$$\begin{aligned}
\psi_0^2 - \Omega^2 &= 0 && : \epsilon^0\\
i\dot{\psi}_0 - 2\psi_0\psi_1 &= 0 && : \epsilon^1\\
i\dot{\psi}_1 - 2\psi_0\psi_2 - \psi_1^2 &= 0 && : \epsilon^2\\
i\dot{\psi}_2 - 2\psi_0\psi_3 - 2\psi_1\psi_2 &= 0 && : \epsilon^3\\
i\dot{\psi}_3 - 2\psi_0\psi_4 - 2\psi_1\psi_3 - \psi_2^2 &= 0 && : \epsilon^4\\
&\vdots
\end{aligned}$$

From this sequence, the general expression for ψ_n is extracted.

[3] Actually, the most desirable solution would have constant amplitude; but it is impossible for such a solution to satisfy (4) regardless of x. Therefore, amplitude variation is included in the form of the solution.

$$\psi_n = \frac{1}{2\psi_0}\left[i\dot{\psi}_{n-1} - \sum_{\substack{i,j\geq 0\\ i+j=n}} a_{ij}\psi_i\psi_j\right], \qquad n > 0,$$

$$a_{ij} = \begin{cases} 2, & i \neq j \\ 1, & i = j \end{cases} \tag{10}$$

$$\psi_0 = \Omega.$$

Substituting (10) into (9) and replacing ψ with $\epsilon\omega$ and Ω with $\epsilon\sqrt{x} \triangleq \epsilon m(t)$, the following expression[4] for ω is obtained.

$$\omega(t) = \sum_{n=0}^{\infty} \omega_n(t)$$

$$\omega_n(t) = \frac{1}{2m(t)}\left[i\dot{\omega}_{n-1} - \sum_{\substack{i,j\geq 0\\ i+j=n}} a_{ij}\omega_i\omega_j\right], \qquad n > 0,$$

$$a_{ij} = \begin{cases} 2, & i \neq j \\ 1, & i = j \end{cases}$$

$$\omega_0 = m(t).$$

Since odd-indexed ω's are imaginary, the desired solution of (4), which is the real part of (6), can be written as[5]

$$y(t) = \exp\left[i\int (\omega_1 + \omega_3 + \omega_5 + \cdots)\, dt\right] \cdot \cos\left[\int (\omega_0 + \omega_2 + \omega_4 + \cdots)\, dt\right]$$

where[6]

$$\omega_0 = m(t)$$

$$\omega_1 = i\frac{1}{2}\frac{\dot{m}}{m}$$

$$\omega_2 = \frac{1}{m}\left[\frac{3}{8}\left(\frac{\dot{m}}{m}\right)^2 - \frac{1}{4}\frac{\ddot{m}}{m}\right]$$

$$\omega_3 = i\frac{1}{m^2}\left[\frac{3}{4}\frac{\dot{m}\ddot{m}}{m^2} - \frac{3}{4}\left(\frac{\dot{m}}{m}\right)^3 - \frac{1}{8}\frac{\overset{(3)}{m}}{m}\right] \tag{11}$$

$$\omega_4 = \frac{1}{m^3}\left[\frac{1}{16}\frac{\overset{(4)}{m}}{m} - \frac{297}{128}\left(\frac{\dot{m}}{m}\right)^4 - \frac{13}{32}\left(\frac{\ddot{m}}{m}\right)^2 - \frac{5}{8}\frac{\dot{m}\overset{(3)}{m}}{m^2} + \frac{99}{32}\frac{(\dot{m})^2\ddot{m}}{m^3}\right]$$

$$\vdots$$

Referring to (3), the following definitions can be made.

$$A(t) \triangleq \exp\left[i\int \sum_{\substack{n>0\\ n\text{ odd}}} \omega_n(t)\, dt\right]$$

$$D(t) \triangleq \sum_{\substack{n>0\\ n\text{ even}}} \omega_n(t)$$

$$KV_{\text{mod}} \triangleq m(t) \triangleq \sqrt{x} \triangleq \text{modulating function.}$$

Case 1A

For the parallel $LC(t)$-type and series $L(t)C$-type VFOs, the desired signal V_{FM} is (from Table I) proportional to xy; but $x = m^2$. Therefore $V_{FM} \propto m^2y(t)$; that is,

$$V_{FM} = B(t)\cos\left[\int (m(t) + D(t))\, dt\right]. \tag{12}$$

$B(t)$ is amplitude modulation, $m(t)$ is the desired modulated frequency, and $D(t)$ is the rate distortion and is given by the following expression:

$$D(t) = \frac{1}{m}\left[\frac{3}{8}\left(\frac{\dot{m}}{m}\right)^2 - \frac{1}{4}\frac{\ddot{m}}{m}\right] + \frac{1}{m^3}\left[\frac{1}{16}\frac{\overset{(4)}{m}}{m} - \frac{297}{128}\left(\frac{\dot{m}}{m}\right)^4 - \frac{13}{32}\left(\frac{\ddot{m}}{m}\right)^2 - \frac{5}{8}\frac{\dot{m}\overset{(3)}{m}}{m^2} + \frac{99}{32}\frac{(\dot{m})^2\ddot{m}}{m^3}\right] + \cdots. \tag{13}$$

Case 1B

For the parallel $L(t)C$-type and series $LC(t)$-type VFOs, the desired signal V_{FM} is (from Table I) proportional to $\int xy(t)\, dt$; but $x = m^2$, therefore $V_{FM} \propto \int m^2y(t)\, dt$; that is,[7]

$$V_{FM} = \int \left(B(t)\cos\left[\int (m(t) + D(t))\, dt\right]\right) dt. \tag{14}$$

This integral is not of a simple form. This complication can be circumvented by making the substitution $y = \dot{z}/x$ before solving (4). The same method of solution is applicable to the transformed differential equation. The desired solution is proportional to the real part of $z(t)$ and is given by

$$V_{FM} = B'(t)\cos\left[\int (m(t) + D'(t))\, dt\right]. \tag{15}$$

The difference between $D'(t)$ in this case and $D(t)$ in Case 1A is contained solely in the numerical coefficients. The form is identical in both cases.

Case 2

The governing differential equation representing the remaining six VFOs is

$$\ddot{y} - \frac{\dot{x}}{x}\dot{y} + x^2y = 0. \tag{16}$$

[4] In this expression, the perturbation parameter ϵ has disappeared.

[5] A general statement about the convergence of this infinite series would present difficulties. However, it can be seen from the expressions in (11) that for a slowly varying modulation function (m) ω_1, ω_2, ω_3, ω_4 grow progressively smaller. This is illustrated with the example in the section "Interpretation of the Solutions." For a rigorous treatment, see [13], [14].

[6] ω_0 and ω_1 comprise what is known as the WKBJ approximation to the solution of (4); this approximation requires $\omega_2 \ll \omega_0$. See [5], [15].

[7] V_{FM} (Case 1A) $= d/dt\,[V_{FM}(\text{Case 1B})]$.

Again, as in Case 1, a method of checking on the plausibility of ideal frequency modulation is to assume the desired solution, substitute it into (16), and solve for the modulation function $x(t)$, which relates the modulating signal V_{mod} to the frequency deviating elements. The desired solution is as shown in (1). The resulting modulation function is

$$x(t) = KV_{\text{mod}}.$$

That is, the exact solution [8], [9] of (16) is

$$y(t) = A_0 \cos \left[\int x(t)\, dt + \theta \right] \tag{17}$$

where A_0 and θ are constants.

Case 2A

For the bridged-T $R(t)C$-type and Wien bridge $R(t)C$-type VFOs, the desired signal V_{FM} is (from Table I) equal to $y(t)$. Therefore,

$$V_{\text{FM}} = A_0 \cos \left[K \int V_{\text{mod}}\, dt + \theta \right] \tag{18}$$

where $V_{\text{mod}} \propto 1/R(t)$. From (18) it is clear that the RC-type VFOs in which the resistance R is modulated inversely proportional to the modulating signal, and the capacitance C is constant, produce ideal frequency modulation with constant amplitude. There is no rate distortion.

Case 2B

For the bridged-T $RC(t)$-type and Wien bridge $RC(t)$-type VFOs and the $L(t)C(t)$-type VFOs, the desired signal V_{FM} is (from Table I) proportional to $x(t)y(t)$. Therefore,

$$V_{\text{FM}} = A_0 V_{\text{mod}} \cos \left[K \int V_{\text{mod}}\, dt + \theta \right] \tag{19}$$

where $V_{\text{mod}} \propto 1/C(t)$ for the RC-type VFOs and $V_{\text{mod}} \propto 1/C(t) \propto 1/L(t)$ for the LC-type VFOs. From (19) it is clear that the RC-type VFOs in which the capacitance is modulated inversely proportional to the modulating signal and the LC-type VFOs in which the inductance and the capacitance are modulated together, each inversely proportional to the modulating signal, produce amplitude product modulation in additional to ideal frequency modulation. There is no rate distortion.

Interpretation of the Solutions

The results of the preceding section may be summarized as follows.

1) The VFOs considered, in which no energy storage elements are modulated, produce ideal frequency modulation with no amplitude modulation and no rate distortion.
2) The VFOs considered, in which both storage elements are modulated proportionally, produce ideal frequency modulation with amplitude product modulation and no rate distortion.
3) The VFOs considered, in which one of the two storage elements is modulated, produce rate distortion and rate-sensitive amplitude modulation.

From these results, it can be concluded that rate distortion must be the result of nontracking energy storage capacities. More specifically, it can be seen from the expression for rate distortion, as shown in (13), that rate distortion increases from zero as the ratio

$$\left(\frac{\text{rate at which the frequency of oscillation is deviated}}{\text{magnitude of the frequency of oscillation}} \right)$$

increases from zero. But the rate at which the frequency of oscillation is deviated is proportional to the rate at which the energy storage capacity of one element varies relative to the storage capacity of the other, and the magnitude of the frequency of oscillation is proportional to the rate at which energy is exchanged between the two storage elements. Therefore, rate distortion increases from zero as the ratio

$$\left(\frac{\text{relative rate of change of storage capacity}}{\text{rate of change of stored energy}} \right)$$

increases from zero.

Thus, rate distortion is sensitive to the ratio of two rates of variation, and as brought out in the development of the solutions for Cases 1A and 1B, rate distortion is sensitive to the type of energy storage element being deviated in a particular configuration.

Example

To exemplify the nature of rate distortion, a parallel $LC(t)$-type VFO [see Fig. 2(a)] with single-frequency modulation will be considered. The modulating signal is given by the following expression:

$$V_{\text{mod}} = V_0(1 + X \cos \omega_m t) \tag{20}$$

where X and ω_m are constants. The modulated signal is given by (12) and the rate distortion by (13), where $m(t) \triangleq \sqrt{x}$. But from Table I,

$$\sqrt{x} = \frac{1}{\sqrt{L_0 C(t)}} = \frac{1}{\sqrt{L_0 C_0}} \cdot \frac{1}{\sqrt{C(t)/C_0}}.$$

If the variable capacitor is a hyperabrupt junction varactor diode with its junction capacitance inversely proportional to the square of the reverse bias voltage [10],

$$C(t) = C_0 \left(\frac{V_{\text{mod}}}{V_0} \right)^{-2}.$$

Therefore,

$$m(t) = \omega_0(1 + X \cos \omega_m t)$$
$$\omega_0 \triangleq \frac{1}{\sqrt{L_0 C_0}}. \tag{21}$$

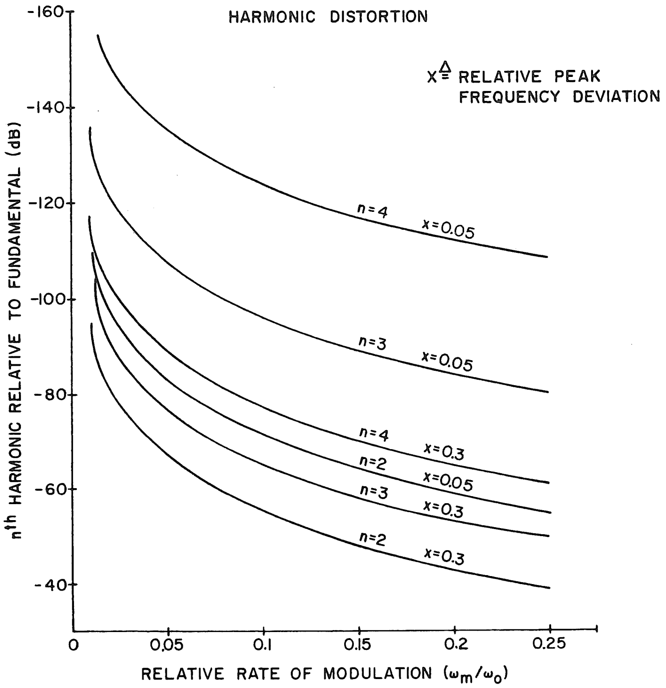


Fig. 3. Harmonic distortion in the FM output of a parallel $LC(t)$-type VFO modulated with $V_{\text{mod}} = V_0\ (1 + X\cos\omega_m t)$.

Substituting (21) into (13) yields the following approximate[8] expression for rate distortion:

$$D(t) = \left(\frac{\omega_m}{\omega_0}\right)^2 \sum_{n=1}^{4} a_n \cos(n\omega_m t)$$

$$a_1 = -\frac{1}{4}X - \frac{3}{32}X^3$$

$$a_2 = \frac{9}{16}X^2$$

$$a_3 = -\frac{21}{32}X^3$$

$$a_4 = \frac{15}{32}X^4.$$

Fig. 3 shows a plot of second, third, and fourth harmonic distortion as a function of the relative rate of modulation (ω_m/ω_0) and the relative peak frequency deviation (X).

From Fig. 3 and the expression for rate distortion, it is clear that the amount of distortion increases as the relative rate of modulation and the relative peak frequency deviation increase.

Summary

An analysis of rate distortion in frequency modulators has been presented. It has been shown that rate distortion results from interaction between the energy exchange between two storage elements and the change of the storage capacities of those elements. It has been shown that certain types of frequency modulators create more rate distortion than others, and that some types are distortion free.

If high rate of modulation, high index of modulation, and low distortion are desired, the RC-type VFOs in which field-effect transistors are used as frequency deviating elements look as promising as previously suggested schemes [11], [12], and appear to be theoretically simpler realizations.

Appendix

Derivation of Differential Equation Representing Bridged-T RC(t)-Type VFO [Fig. 2(i)]

Letting[9] $F(V_{FM}) = \frac{3}{2}V_{FM}$ and summing currents into the node between the two capacitors results in

$$\frac{5}{2}\frac{d}{dt}(CV_{FM}) - 2\frac{d}{dt}(CV_0) - \frac{V_0}{R} = 0. \qquad (25)$$

Summing currents into the output node results in

$$\frac{d}{dt}(CV_{FM}) - \frac{d}{dt}(CV_0) - \frac{V_{FM}}{2R} = 0. \qquad (26)$$

Solving (26) for V_0 and $d/dt(CV_0)$ and substituting into (25) results in

$$C\frac{d}{dt}(CV_{FM}) + \frac{1}{R^2}\int V_{FM}\,dt = 0. \qquad (27)$$

Differentiating (27) yields

$$\frac{d^2}{dt^2}(CV_{FM}) + \frac{1}{C}\frac{dC}{dt}\frac{d}{dt}(CV_{FM}) + \frac{(CV_{FM})}{R^2C^2} = 0. \quad (28)$$

Equation (28) may be rewritten as

$$\ddot{y} - \frac{\dot{x}}{x}\dot{y} + x^2 y = 0$$

$$y = C(t)V_{FM}, \qquad x = (RC(t))^{-1}. \qquad (29)$$

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[8] It has been assumed that ω_m/ω_0 and X are small enough that the expression for rate distortion (13) may be truncated after the first term (for $\omega_m/\omega_0 < 1/10$ and $X < 1/3$, the second term is down more than 40 dB from the first term).

[9] This value of gain exactly cancels the losses in the RC section so that the poles of the loop transfer function will be on the imaginary axis.

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