

The Role of Spectral Correlation in Design and Performance Analysis of Synchronizers

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Abstract—The popular class of synchronizers that consist of a quadratic nonlinearity followed by a phase-lock loop is investigated, and it is shown that the optimum design of the quadratic transformation is characterized in terms of a spectral correlation function for the signal to be synchronized to. It is also shown that the SNR performance of this quadratic transformation, and the mean-square phase jitter of the phase-lock loop are both characterized in terms of spectral correlation functions. The conditions under which the optimum quadratic transformations, for symbol synchronization of BPSK, QPSK, SQPSK, and MSK, and for carrier synchronization of BPSK, reduce to the well-known matched-filter-squarer are identified. In addition, the well-known zero-mean-square-phase-jitter condition is generalized from PAM to all synchronizable signals, and is characterized in terms of the spectral correlation function. The low-SNR maximum-likelihood synchronizer for all quadratically synchronizable signals is characterized in terms of a multiplicity of maximum-SNR quadratic spectral-line generators. A closed form implementation in terms of a matched filter, squarer, and symbol-rate-synchronized averager is obtained for BPSK and QPSK signals.

I. INTRODUCTION

THE problem of synchronizing the phase and frequency of an analog oscillator or a digital clock to periodicity contained in a time series is a fundamental problem in many areas of time-series analysis for periodic phenomena, and is especially important for the purposes of sampling, demodulating, demultiplexing, and decoding in communication systems. Because of its crucial role in the proper operation of all coherent communication systems, the synchronization problem has been the subject of considerable research and development for several decades, and especially in recent years with the increasing demands on system performance.

A popular approach to synchronization is to interpret the synchronization problem as consisting of two tasks as illustrated by the block-diagram shown in Fig. 1: i) the task of regenerating a spectral line (which has been annihilated by modulation with random data) by using a nonlinear transformation of the time series to be synchronized to, say $x(t)$, into a timing wave, say $w(t)$; and ii) the task of locking the oscillator or clock frequency and phase to that of the sinewave contained in the timing wave $w(t)$ by using a *phase-lock loop* (PLL) or other similar device such as a bandpass filter and zero-crossing detector. The purpose of this paper is to show that both design and performance analysis of such synchronizers can be characterized in terms of a spectral correlation function.

In Section II, the spectral correlation function for signals that exhibit cyclostationarity is introduced. Then in Section

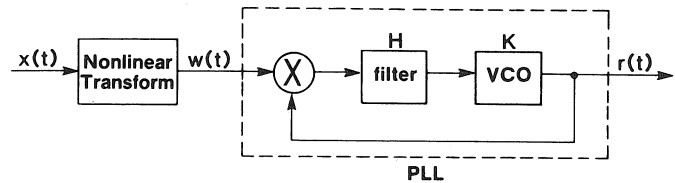


Fig. 1. A general structural form for synchronizers.

III, it is shown that the maximum-SNR quadratic transformation for generation of a spectral line is specified in terms of the spectral correlation function for the signal to be synchronized to and the power spectral density (PSD) of the additive noise. This general solution is studied for various signal types including BPSK, QPSK, SQPSK, and MSK. In Section IV, it is shown that the mean-square phase-jitter of the PLL is characterized in terms of the power in the spectral line from the timing wave $w(t)$, and the spectral correlation function and PSD for the residual timing wave denoted by $n(t)$. Then it is shown that the condition for zero mean-square phase jitter can be specified in terms of a balance between the spectral correlation function and the PSD of $n(t)$. Finally, in Section V, the low-SNR maximum-likelihood synchronizer for all quadratically synchronizable signals is characterized in terms of spectral correlation. It is shown that this synchronizer can be interpreted as a multiplicity of maximum-SNR spectral-line generators. A closed form implementation in terms of a matched filter, squarer, and symbol-rate-synchronized averager is obtained for BPSK and QPSK signals.

The contribution of this paper is not the derivation of new synchronizer designs, nor new evaluations of synchronizer performance, but rather the development of a unifying conceptual framework for the study of synchronization.

II. SPECTRAL CORRELATION¹

A time series $x(t)$ is said to *exhibit cyclostationarity* (in the wide sense) with *cycle frequency* α if and only if the statistical function

$$\hat{R}_x^\alpha(\tau) \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t + \tau/2)x(t - \tau/2)e^{-i2\pi\alpha t} dt, \quad (1)$$

which represents the coefficient of the complex sine wave component of frequency α contained in the lag-product waveform $x(t + \tau/2)x(t - \tau/2)$ for each lag value τ , is not identically zero (as a function of the lag variable τ). A *time-variant autocorrelation function* for such a time series can be defined by

$$\hat{R}_x(t, \tau) = \hat{R}_x^0(\tau) + \sum_T [\hat{R}_x(t, \tau; T) - \hat{R}_x^0(\tau)] \quad (2a)$$

¹ The definitions, terminology, and results presented in this background section are taken from [1] and [2].

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where

$$\hat{R}_x(t, \tau; T) \triangleq \lim_{N \rightarrow \infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^N x(t+nT+\tau/2)x(t+nT-\tau/2) \quad (2b)$$

and the sum in (2a) extends over all incommensurate periods T for which (2b) is not identical to $\hat{R}_x^0(\tau)$. Furthermore, this autocorrelation function can be expanded in a Fourier series, and the Fourier coefficients (which depend on τ) turn out to be the statistical function $\hat{R}_x^\alpha(\tau)$ for α equal to the harmonics (integer multiples) of the various fundamental frequencies $1/T$,

$$\hat{R}_x(t, \tau) = \sum_{\alpha} \hat{R}_x^\alpha(\tau) e^{i2\pi\alpha t}. \quad (3)$$

The statistical function $\hat{R}_x^\alpha(\tau)$ is called the *cyclic autocorrelation function*. For $\alpha = 0$, it reduces to the conventional autocorrelation function denoted by $\hat{R}_x(\tau)$,

$$\hat{R}_x(\tau) \equiv \hat{R}_x^0(\tau). \quad (4)$$

If $x(t)$ exhibits no cyclostationarity, then (3) contains only the $\alpha = 0$ term. If $x(t)$ exhibits cyclostationarity with only one period T , then the sum in (2a) contains only one term, and the sum in (3) extends over only the harmonics of $1/T$.

The Fourier transform of $\hat{R}_x^\alpha(\tau)$ is the conventional PSD,

$$\hat{S}_x(f) = \int_{-\infty}^{\infty} \hat{R}_x(\tau) e^{-i2\pi f\tau} d\tau, \quad (5)$$

which is the density (in f) of time-averaged power (or temporal mean square) of the spectral components of $x(t)$. The Fourier transform of $\hat{R}_x^\alpha(\tau)$,

$$\hat{S}_x^\alpha(f) = \int_{-\infty}^{\infty} \hat{R}_x^\alpha(\tau) e^{-i2\pi f\tau} d\tau, \quad (6)$$

is called the *cyclic spectral density*. It can be shown that this statistical function is a spectral correlation function. That is, $\hat{S}_x^\alpha(f)$ is the density (in f) of correlation between spectral components of $x(t)$ at frequencies $f + \alpha/2$ and $f - \alpha/2$. Specifically,

$$\hat{S}_x^\alpha(f) = \lim_{\Delta f \rightarrow 0} \lim_{\Delta t \rightarrow \infty} \frac{\Delta f}{\Delta t} \cdot \int_{-\Delta t/2}^{\Delta t/2} X_{\Delta f}(t, f + \alpha/2) X_{\Delta f}^*(t, f - \alpha/2) dt \quad (7)$$

where

$$X_{\Delta f}(t, f) = \int_{t-1/2\Delta f}^{t+1/2\Delta f} x(u) e^{-i2\pi fu} du. \quad (8)$$

The limit $\Delta t \rightarrow \infty$ provides the idealized temporal correlation of the two spectral components, and the limit $\Delta f \rightarrow 0$ provides infinitesimal spectral resolution for these two components. For $\alpha = 0$, (7) reduces to the well-known result that the PSD can be obtained from the formula

$$\hat{S}_x(f) = \lim_{\Delta f \rightarrow 0} \lim_{\Delta t \rightarrow \infty} \frac{\Delta f}{\Delta t} \int_{-\Delta t/2}^{\Delta t/2} |X_{\Delta f}(t, f)|^2 dt \quad (9)$$

and this is consistent with the fact that $\hat{S}_x(f)$ is a mean square measure of spectral content, and correlation reduces to mean square when the quantities being correlated are identical ($f + \alpha/2 = f - \alpha/2 \leftrightarrow \alpha = 0$).

The spectral correlation function can be converted to a complex-valued *spectral correlation coefficient* (for each value of f and α) by normalization with the mean square values of the spectral components (the mean values are zero),

$$\hat{C}_x^\alpha(f) = \frac{\hat{S}_x^\alpha(f)}{[\hat{S}_x(f + \alpha/2)\hat{S}_x(f - \alpha/2)]^{1/2}}. \quad (10)$$

It can be verified that

$$|\hat{C}_x^\alpha(f)| \leq 1. \quad (11)$$

The time series $x(t)$ is said to be *completely coherent* with cycle frequency α and spectrum frequency f if

$$|\hat{C}_x^\alpha(f)| = 1. \quad (12)$$

Although the probabilistic counterpart [2] to the theory from [1] summarized here is more in line with the conventional approach to analysis of random signals, this theory based on time-averages is, in the author's opinion, more appropriate for the study of synchronization, where, for example, one would like mean-square phase jitter to correspond to the amount of phase jitter over time, not over an ensemble.

Examples of explicit formulas for the spectral correlation function for BPSK, QPSK, SQPSK, and MSK are given in the Appendix, and their graphs are shown in Figs. 2-5.

III. OPTIMUM SPECTRAL-LINE GENERATION

The general input-output relation for a quadratic time-invariant (QTI) transformation is

$$w_\alpha(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_\alpha(u, v) x(t-u) x(t-v) du dv \quad (13)$$

where k_α is a weighting function (kernel) analogous to the impulse-response function for a linear time-invariant transformation. The subscript α denotes the frequency at which it is desired to generate a spectral line (e.g., for BPSK, the symbol rate or twice the carrier frequency). The strength of the spectral line at frequency α in $w_\alpha(t)$ is given by²

$$P_w^\alpha = \left| \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} w_\alpha(t) e^{-i2\pi\alpha t} dt \right|^2. \quad (14)$$

Substitution of (13) into (14), and use of definitions (1) and (5) and Parseval's relation for Fourier transforms, yields the formula

$$P_w^\alpha = \left| \int_{-\infty}^{\infty} K_\alpha(f + \alpha/2, f - \alpha/2) \hat{S}_x^\alpha(f) df \right|^2, \quad \alpha \neq 0 \quad (15)$$

where K_α is the double Fourier transform of k_α (analogous to the transfer function for a linear time-invariant transformation),

$$K_\alpha(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_\alpha(u, v) e^{-i2\pi(\mu u - \nu v)} du dv. \quad (16)$$

In obtaining (15), it was assumed that $x(t)$ consists of a signal s plus additive stationary noise m ,

$$x(t) = s(t) + m(t), \quad (17)$$

in which case $\hat{S}_x^\alpha(f) = \hat{S}_s^\alpha(f)$ for $\alpha \neq 0$.

The power spectral density of the timing wave $w_\alpha(t)$ at the

² Strictly speaking, this requires temporal mean square (obtained by replacing t with $t + u$ in (14) and averaging over all u) convergence in (14); cf. [3].

output of the QTI transformation, when there is only stationary Gaussian noise $x(t) = m(t)$ at the input, can be obtained from (13) with the use of definitions (1), (4), and (9). The result is [2]

$$\hat{S}_w(f) = 2 \int_{-\infty}^{\infty} |K_\alpha(\nu + f/2, \nu - f/2)|^2 \hat{S}_m(\nu + f/2) \cdot \hat{S}_m(\nu - f/2) d\nu, \quad f \neq 0. \quad (18)$$

A measure of SNR for the timing wave $w_\alpha(t)$ is

$$\text{SNR}^\alpha = \frac{P_w^\alpha}{B \hat{S}_w(\alpha)} \quad (19)$$

where B is the output bandwidth of the quadratic device. Although this measure is easiest to justify for low SNR applications, such as satellite communications and spread-spectrum signal-interception (because it ignores the signal component in $w_\alpha(t)$ that has continuous PSD, and it ignores the signal-cross-noise terms—all of which are negligible for sufficiently low SNR), it is more generally appropriate for purposes of design, but not necessarily for performance evaluation.³

After substitution of (15) and (18) into (19), the Cauchy-Schwarz inequality can be used to show that SNR^α is maximized if and only if

$$K_\alpha(f + \alpha/2, f - \alpha/2) = \frac{c \hat{S}_s^\alpha(f)^*}{\hat{S}_m(f + \alpha/2) \hat{S}_m(f - \alpha/2)} \quad (20)$$

for any nonzero constant c . The resultant maximum value of SNR^α is given by the formula

$$\text{SNR}_{\max}^\alpha = \frac{1}{2B} \int_{-\infty}^{\infty} \frac{|\hat{S}_s^\alpha(f)|^2}{\hat{S}_m(f + \alpha/2) \hat{S}_m(f - \alpha/2)} df. \quad (21)$$

It follows from (20) and (21) that the optimum quadratic transformation for maximum-SNR generation of a spectral line at frequency α is completely specified by the spectral correlation function \hat{S}_s^α for the signal and the PSD \hat{S}_m for the noise. Explicit formulas for the spectral correlation function have been calculated for general models of various digital signal formats, including BPSK, QPSK, SQPSK, and MSK [1]–[3], and simplified versions for specific models are summarized in the Appendix. The formulas for \hat{S}_s^α reveal how the optimum QTI transformation depends on the particular modulation format as well as the pulse-shape and data correlation. For example, for the simplest case in which the data are uncorrelated (white), it can be shown [using the factorization (A2b), (A2c)] that for BPSK [given by (A1)] the optimum QTI transformation, as specified by (13), (16), (20), and the formula (A2a) for $\hat{S}_s^\alpha(f)$, is exactly implemented by a matched filter (matched to the carrier burst) followed by a squarer, and this is true for both α equal to twice the carrier frequency $2f_c$ and α equal to the symbol rate f_s . In fact, the same device is optimum for $\alpha = kf_s$ and $\alpha = 2f_c + kf_s$ for all integers k . The transfer function of the matched filter is given by

$$H_\alpha(f) = \frac{1}{\hat{S}_n(f)} [Q(f + f_c) e^{i(2\pi f_c t_0 + \phi_0)} + Q(f - f_c) e^{-i(2\pi f_c t_0 + \phi_0)}]$$

where $Q(f)$ is the sinc function (A2b), and t_0 and ϕ_0 are the symbol timing and carrier phase parameters [cf. (A1)]. For

³ This same measure of performance is used in [4] for prefilter design for the delay-and-multiply quadratic synchronizer.

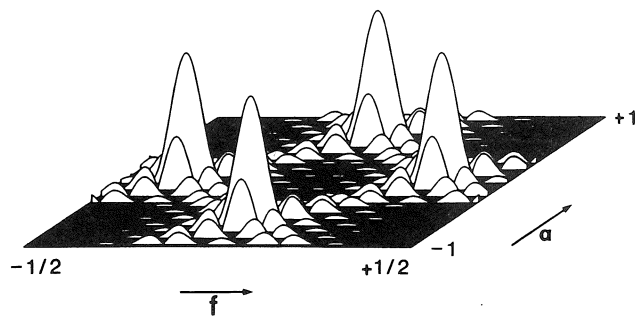


Fig. 2. Spectral correlation magnitude for BPSK.

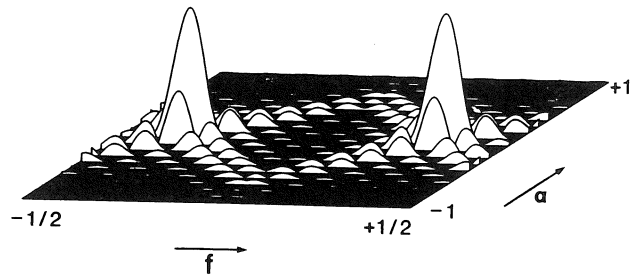


Fig. 3. Spectral correlation magnitude for QPSK.

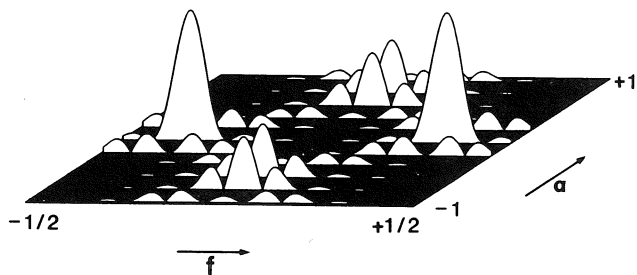


Fig. 4. Spectral correlation magnitude for SQPSK.

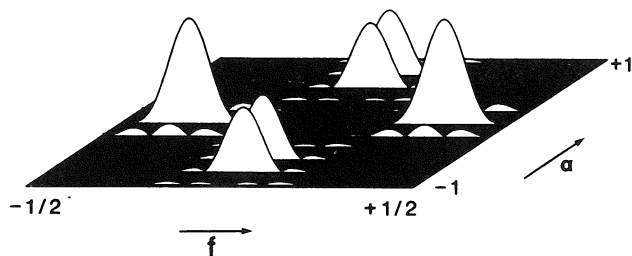


Fig. 5. Spectral correlation magnitude for MSK.

example, for white noise ($\hat{S}_n(f) = N_0$), the impulse response is given by

$$h_\alpha(t) = \frac{2}{N_0} \cos(2\pi f_c(t - [t_0 + \phi_0/f_c])), \quad |t| \leq T_s/2,$$

a delayed causal version of which can be implemented, except for the unknown phase parameter $t_0 + \phi_0/f_c$. The timing parameter estimates can be obtained from the timing parameters (pulse-timing and carrier phase) of the timing wave $w_\alpha(t)$. Use of an arbitrary phase in place of $t_0 + \phi_0/f_c$ will not degrade the SNR performance. For QPSK (with in-phase and quadrature data that are both white and uncorrelated with each

other), it can be shown [using the factorization (A4b), (A4c), and (A2c)] that the optimum QTI transformation is exactly implemented by a pair of in-phase and quadrature matched filters, each followed by a squarer, the outputs of which are summed. In fact, this same device is optimum for $\alpha = kf_s$ for all integers k . For SQPSK, the device that is optimum for BPSK is optimum for SQPSK for $\alpha = 2f_c + kf_s$ for only odd integers k , and the device that is optimum for QPSK is optimum for SQPSK for $\alpha = kf_s$ for only even integers k . No other spectral lines can be generated (with a QTI transformation) from balanced SQPSK. The same synchronizer is optimum for MSK, except that the filter is matched to the carrier burst with the modified (half cosine) envelope used for MSK.

IV. PLL PERFORMANCE

The phase-tracking performance of a PLL following *any* nonlinear spectral line generating device is explicitly characterized in terms of the spectral correlation and PSD of the timing wave $w(t)$, including the power P of the spectral line. Specifically, consider the decomposition of the timing wave into the spectral line of interest and a residual, say $n(t)$,

$$w(t) = (2P)^{1/2} \cos(2\pi\alpha t - \phi) + n(t). \quad (22)$$

The reference waveform in the PLL shown in Fig. 1 is

$$r(t) = \sin[2\pi\beta t - \theta - K\psi(t)] \quad (23)$$

where K is the VCO gain, and $\psi(t)$ is the output of the phase-detector/loop-filter

$$\psi(t) = \int_{-\infty}^t [w(u)r(u)] \otimes h(u) du \quad (24)$$

(where \otimes denotes convolution). Since the loop filter, with impulse-response function $h(t)$, has bandwidth B [not the same as B in (19)] less than α , say (ideally)

$$H(f) = 0, \quad |f| \geq B < \alpha, \quad (25)$$

then the sum-frequency ($\alpha + \beta$) term in (24) is negligible, and (24) yields

$$d\psi(t)/dt = \{(P/2)^{1/2} \sin[e(t)] + n(t)\sin[2\pi\alpha t - \phi - e(t)]\} \otimes h(t), \quad (26)$$

where $e(t)$ is the loop tracking error

$$e(t) \triangleq 2\pi(\alpha - \beta)t + \theta + K\psi(t) - \phi. \quad (27)$$

If it is assumed that the loop is tracking so that $|e(t)| \ll 1$ [in which case $\sin[e(t)] \cong e(t)$ and $\sin[2\pi\alpha t - \phi - e(t)] \cong \sin(2\pi\alpha t - \phi)$], and if it is assumed that the product of the power P and the VCO gain K is sufficiently large that

$$\frac{H(f)}{\sqrt{P/2} H(f) - i2\pi f/K} \cong \frac{1}{\sqrt{P/2}}, \quad |f| < B \quad (28)$$

is a close approximation, then (26) and (27) yield the close approximation

$$e(t) \cong (2/P)^{1/2} \{n(t)\sin(2\pi\alpha t - \phi)\} \otimes h_0(t) + 2\pi(\alpha - \beta)/K(P/2)^{1/2} \quad (29)$$

where h_0 is the impulse-response function of an ideal LPF with bandwidth B . If it is assumed that the constant term in (29) is negligible, then it follows from the formula for the PSD of a cyclostationary AM waveform put through a filter [1], [2] that

the mean (time-averaged) squared value of this phase error is given by⁴

$$\text{MSE} = (1/2P) \int_{-B}^B [\hat{S}_n(f+\alpha) + \hat{S}_n(f-\alpha) - \hat{S}_n^{2\alpha}(f)e^{i2\phi} - \hat{S}_n^{-2\alpha}(f)e^{-i2\phi}] df. \quad (30)$$

Thus, the only factors that determine how small this MSE is are

- i) the power P of the generated spectral line in the timing wave $w(t)$, and
- ii) the imbalance, within the passband of the loop filter, between the frequency-shifted spectrum and the phase-shifted spectral correlation function of the residual $n(t)$ in the timing wave,

$$[\hat{S}_n(f+\alpha) + \hat{S}_n(f-\alpha)] - [\hat{S}_n^{2\alpha}(f)e^{i2\phi} + \hat{S}_n^{-2\alpha}(f)e^{-i2\phi}], \quad |f| < B < \alpha. \quad (31)$$

It can be shown [2], by using Rice's representation

$$n(t) = c(t)\cos(2\pi\alpha t - \phi) - s(t)\sin(2\pi\alpha t - \phi) \quad (32)$$

for the residual $n(t)$, that

$$\frac{1}{2} [\hat{S}_c(f) + \hat{S}_s(f)] = \hat{S}_n(f+\alpha) + \hat{S}_n(f-\alpha), \quad |f| < B < \alpha \quad (33a)$$

and

$$\frac{1}{2} [\hat{S}_c(f) - \hat{S}_s(f)] = \hat{S}_n^{2\alpha}(f)e^{i2\phi} + \hat{S}_n^{-2\alpha}(f)e^{-i2\phi}, \quad |f| < B < \alpha. \quad (33b)$$

This reveals that the expression in (31), the integrand of (30), is simply the PSD $\hat{S}_s(f)$ of the quadrature component of the residual. This corroborates the known fact that the synchronization error is due only to the quadrature component $s(t)$ in (32). An alternative approach to interpreting expression (31) is developed in the following.

The effect on MSE of an imbalance between the two terms in (31) cannot be reduced simply by reduction of the filter bandwidth B because this degrades acquisition and dynamic tracking performance. The significance of the condition of a balance between the two terms in (31), which yields zero MSE, can be understood both statistically and deterministically. Specifically, consider the condition that the residual $n(t)$ have zero-crossings that coincide with those of the sinusoidal component of $w(t)$,

$$n(kT_0 + t_0) = 0 \quad (34a)$$

for all integers k , where

$$T_0 \triangleq 1/2\alpha \quad (34b)$$

$$t_0 \triangleq 1/4\alpha + \phi/2\pi\alpha. \quad (34c)$$

We shall see that if $n(t)$ is band limited to $f \in [-2\alpha, 2\alpha]$, then (34) is sufficient (but not necessary) for a balance between the terms in (31), and furthermore, that (34) is sufficient for $n(t)$ to be completely coherent [cf. (12)] for $|f|$

⁴ This result corroborates that in [6], except that \hat{S}_n^α is not recognized as the spectral correlation function in [6]. It is worth noting that when $n(t)$ is stationary ($\hat{S}_n^\alpha(f) = 0$ for all $\alpha \neq 0$) and the loop bandwidth B is small, then MSE is proportional to the reciprocal of SNR^a, with proportionality constant determined by the ratio of bandwidths in (19) and (30).

$\leq \alpha$, because (34) is necessary and sufficient for a band-limited $n(t)$ to be an AM waveform, which is known to be completely coherent [1], [2].

The condition (34) is equivalent to

$$y(t) \triangleq n(t) \sum_k \delta(t - t_0 - kT_0) \equiv 0 \quad (35)$$

which is equivalent (by the Poisson sum formula [7]) to

$$y(t) = n(t) \sum_m e^{i2\pi m(t-t_0)/T_0} \equiv 0. \quad (36)$$

It follows that $\hat{S}_y(f) \equiv 0$, from which (36) (using the formula for the PSD of generalized AM [1], [2]) yields

$$\sum_{p,m} \hat{S}_n^{(p-m)/T_0}(f + (p+m)/2T_0) e^{i2\pi(p-m)t_0/T_0} \equiv 0 \quad (37)$$

which is necessary, but not sufficient, for (34). In practice, the timing wave $w(t)$, and therefore its residual $n(t)$, is band limited to $f \in [-2\alpha, 2\alpha]$. Consequently, (37) reduces to a balance between the terms in (31), which is necessary and sufficient for $\text{MSE} = 0$. However, (34) guarantees that the zero-crossings of the timing wave are precisely periodic, and that the so-called *phase jitter* is therefore zero in a deterministic sense, which we see is sufficient but not necessary for the mean square phase jitter to be zero. To obtain a spectral characterization of the *zero-phase-jitter condition* (34), which is both necessary and sufficient, the fact that $Y(f)$, the Fourier transform of $y(t)$ (treated as a finite-energy function) in (36), is identically zero,

$$Y(f) = \sum_m N(f - m/T_0) e^{-i2\pi m t_0/T_0} \equiv 0, \quad (38)$$

can be used. If $n(t)$ is band limited to $f \in [-1/T_0, 1/T_0]$, then only the $m = \pm 1$ terms in (38) are nonzero, and (38) can therefore be reexpressed as the following necessary and sufficient condition:

$$N(f) + N(f - 1/T_0) e^{-i2\pi t_0/T_0} + N(f + 1/T_0) e^{i2\pi t_0/T_0} \equiv 0, \quad |f| \leq 1/T_0$$

which in turn can be expressed as

$$\{n(t) + 2n(t)\cos[2\pi(t-t_0)/T_0]\} \otimes \frac{\sin(2\pi t/T_0)}{2\pi t/T_0} \equiv 0. \quad (39)$$

Substitution of Rice's representation (32), with $\alpha = 1/2T_0$, $\phi = \pi t_0/T_0$, and $c(t)$ and $s(t)$ band limited to $f \in (-1/2T_0, 1/2T_0)$, into (39) yields $c(t) \equiv 0$, in which case (32) becomes

$$n(t) = -s(t)\sin[\pi(t-t_0)/T_0]$$

which can be reexpressed as [using (34b) and (34c)]

$$n(t) = s(t)\cos(2\pi\alpha t - \phi). \quad (40)$$

Thus, a timing wave band limited to $f \in [-2\alpha, 2\alpha]$ satisfies the zero-phase-jitter condition if and only if it is an AM waveform, in which the quadrature component is identically zero. It can be shown that such an AM waveform satisfies the *zero-mean-square-phase-jitter condition* [which is a balance between the two terms in (31)], but other types of waveforms also can satisfy this condition, e.g., any waveform that differs from the AM waveform (40) by any waveform with zero average power (power averaged over all time).

Application of these results to a PAM time series reveals that the known symmetry condition, on the prefilter of a synchronizer, that guarantees zero phase jitter [8] guarantees that the PAM waveform is transformed into an AM waveform. Similarly, the known symmetry condition on the postfilter guarantees that the AM property is preserved (no AM-to-PM conversion).

V. LOW-SNR MAXIMUM-LIKELIHOOD SYNCHRONIZERS

It is well known that a monotonic function of the likelihood ratio, for a weak zero-mean signal in additive white Gaussian noise on the time interval $[t - T/2, t + T/2]$, is closely approximated by the quadratic form [9]

$$y(t) = \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \hat{R}_s(t - [u+v]/2, u-v) \cdot x(t-u)x(t-v) du dv \quad (41)$$

where $\hat{R}_s(t, \tau)$ is the time-variant autocorrelation function (2)–(3) for the signal (cf. [3]). Consequently, the low-SNR maximum-likelihood (ML) estimates of any unknown timing parameters, on which $\hat{R}_s(t, \tau)$ is dependent, can be determined from maximization of $y(t)$ with respect to these parameters. To relate this low-SNR ML synchronizer to the maximum-SNR spectral-line generator, the representation (3) can be substituted into (41), and the result can be manipulated into the form [2]

$$y(t) = \sum_{\alpha} \int_{-\infty}^{\infty} \hat{S}_s^{\alpha}(f) * S_{xT}^{\alpha}(t, f) df \quad (42)$$

where S_{xT}^{α} is the *cyclic periodogram* [cf. (7), (8)] defined by

$$S_{xT}^{\alpha}(t, f) = \frac{1}{T} X_{1/T}(t, f + \alpha/2) X_{1/T}^*(t, f - \alpha/2). \quad (43)$$

If the maximum-SNR spectral-line-generating QTI transformation, specified by (13), (16), and (20), for white noise $\hat{S}_m(f) \equiv N_0$, is modified by the addition of an output narrow bandpass filter centered at frequency α , with bandwidth $1/T$, then it can be shown⁵ [5] that the output of the spectral line generator is closely approximated by

$$y_{\alpha}(t) \approx 2 \operatorname{Re} \left\{ \int_{-\infty}^{\infty} \hat{S}_s^{\alpha}(f) * S_{xT}^{\alpha}(t, f) df e^{i2\pi\alpha t} \right\}. \quad (44)$$

It follows that the low-SNR ML synchronizer (42) is simply a multiplicity of real maximum-SNR spectral-line generators, with outputs down-converted to baseband and summed. For example, for BPSK each spectral-line generator is a matched-filter-squarer followed by a narrow bandpass filter. Although this characterization of the low-SNR ML synchronizer is conceptually valuable, it does not necessarily suggest a useful approach to implementation. For example, for BPSK the outputs of the matched filter-squarers must be phase-compensated [a consequence of the factor $e^{-i2\pi\alpha t_0}$ in (A2b)] before down-conversion, and the timing parameter t_0 required for this is unknown. Furthermore, for the signals considered so far, namely BPSK, QPSK, SQPSK, MSK, and APK, the closed-form formula (41) leads directly to potentially attractive implementations. For example, for BPSK with carrier frequency f_c , carrier phase ϕ_0 , symbol rate f_s , symbol timing t_0 ,

⁵ It should be pointed out that the synchronization problem is not considered in [5], but the QTI transformations that are optimum for low-SNR synchronization [as specified by (20) and (42)] are also optimum for low-SNR detection, which is the topic investigated in [5].

and with white data [cf. (A1)], we have [2]

$$\begin{aligned} \hat{R}_s(t - [u + v]/2, u - v) = & \cos[2\pi f_c(t - u) + \phi_0] \\ & \cdot \cos[2\pi f_c(t - v) + \phi_0] \\ & \cdot \sum_{n=-\infty}^{\infty} q(t - t_0 - u - nT_s) \\ & \cdot q(t - t_0 - v - nT_s). \end{aligned} \quad (45)$$

Substitution of (45) into (41) yields

$$y(t) = \sum_{n=-\infty}^{\infty} \left[\int_{t-T/2}^{t+T/2} q(u - t_0 - nT_s) \cdot \cos(2\pi f_c u + \phi_0) x(u) du \right]^2 \quad (46)$$

$$\begin{aligned} & \equiv \sum_{n=-N}^N \left[\int_{-\infty}^{\infty} q(u - k_i T_s - nT_s) \cdot \cos(2\pi f_c u + \phi_0) x(u) du \right]^2 \end{aligned} \quad (47)$$

where k_i is the closest integer to $(t - t_0)/T_s$, and N is the closest integer to $(1/2)(T/T_s - 1)$. It follows from (47) that the ML synchronizer can be implemented (to a close approximation for $T \gg T_s$) as a synchronous product demodulator, followed by a filter matched to the symbol envelope $q(t)$, followed by a square-law device and a sliding *symbol-rate-synchronized averaging device* that superposes $2N + 1$ adjacent waveform segments of length T_s , and adds them together; finally, the averaged waveform is sampled at precisely the right time $k_i T_s$, which does not slide along continuously with t , but rather jumps by an amount T_s once every T_s units of time. Since the clock timing t_0 and carrier phase ϕ_0 are not known, then the synchronous product demodulator, matched filter, and sampler cannot be implemented. However, use of some value other than t_0 in the matched filter only shifts the location $k_i T_s$ of the periodically occurring peak in the output waveform $y(t)$, but ϕ_0 must be adaptively adjusted via some feedback mechanism operating on the timing wave $y(t)$.

It should be clarified that if the received signal $x(t)$ is band limited to twice the Nyquist (zero intersymbol interference) bandwidth for BPSK (i.e., 100 percent excess bandwidth), then only the terms in (42) corresponding to the symbol rate and twice the carrier frequency are nonzero. None of the harmonics of the symbol rate remain. In this case, the symbol-synchronized averaging device can be replaced with a narrow bandpass filter at either $2f_c$ for carrier synchronization or at f_s for symbol synchronization. Thus, the low-SNR ML synchronizers for signals with less than 100 percent excess bandwidth are identical to the maximum-SNR spectral line generators described in Section III. This is in agreement with the ML synchronizers derived in [10].

VI. CONCLUSIONS

It has been shown that wide-sense cyclostationarity is characterized by a spectral correlation function, and it has been shown that the design of quadratic synchronizers is conveniently characterized by the spectral correlation function. This applies to both the maximum-SNR design criterion and the ML design criterion for weak signals. It has also been shown that the SNR-performances of quadratic spectral-line generators and the mean-square phase-jitter performance for phase-lock loops are conveniently characterized by the spec-

tral correlation function. Consequently, the theory of spectral correlation for time series that exhibit cyclostationarity [1]–[3] provides a unifying framework for the study of synchronization.

APPENDIX

SPECTRAL CORRELATION FUNCTIONS

BPSK: A BPSK signal can be modeled by

$$x(t) = a(t) \cos(2\pi f_c t + \phi_0) \quad (A1a)$$

where

$$a(t) = \sum_{n=-\infty}^{\infty} a_n q(t - t_0 - nT_s) \quad (A1b)$$

in which $q(t)$ is a rectangle pulse with height of unity and width of T_s , and $\{a_n\}$ is a binary (± 1) sequence. If $\{a_n\}$ is modeled as a sequence of independent variables with equiprobable values, then it can be shown [2], [3] that the spectral correlation function is given by

$$\begin{aligned} \hat{S}_x^\alpha(f) = & \frac{1}{4T_s} [Q(f + f_c + \alpha/2)Q(f + f_c - \alpha/2) \\ & + Q(f - f_c + \alpha/2)Q(f - f_c - \alpha/2)] e^{-i2\pi\alpha t_0} \\ & + \frac{1}{4T_s} [Q(f + f_c + \alpha/2) \\ & \cdot Q(f - f_c - \alpha/2) e^{-i(2\pi[\alpha + 2f_c]t_0 + 2\phi_0)} \\ & + Q(f - f_c + \alpha/2) \\ & \cdot Q(f + f_c - \alpha/2) e^{-i(2\pi[\alpha - 2f_c]t_0 - 2\phi_0)}] \end{aligned} \quad (A2a)$$

for $\alpha = \pm 2f_c + k/T_s$ and $\alpha = k/T_s$ for all integers k , and $\hat{S}_x^\alpha(f) \equiv 0$ for all other values of α . In (A2a), $Q(f)$ is the sinc function

$$Q(f) = \frac{\sin(\pi f T_s)}{\pi f}.$$

Equation (A2a) can be factored into the form

$$\hat{S}_x^\alpha(f) = K(f + \alpha/2)K(\alpha/2 - f) e^{-i2\pi\alpha t_0}, \quad (A2b)$$

where

$$K(f) \triangleq Q(f + f_c) e^{-i(2\pi f_c t_0 + \phi_0)} + Q(f - f_c) e^{i(2\pi f_c t_0 + \phi_0)}. \quad (A2c)$$

QPSK: A QPSK signal can be modeled by

$$x(t) = c(t) \cos(2\pi f_c t + \phi_0) - s(t) \sin(2\pi f_c t + \phi_0) \quad (A3a)$$

in which

$$\begin{aligned} c(t) &= \sum_{n=-\infty}^{\infty} c_n q(t - t_0 - nT_s) \\ s(t) &= \sum_{n=-\infty}^{\infty} s_n q(t - t_0 - nT_s). \end{aligned} \quad (A3b)$$

If the two binary (± 1) sequences $\{c_n\}$ and $\{s_n\}$ are modeled as independent of each other, and consisting of independent variables with equiprobable values, then it can be shown [2],

[3] that the spectral correlation function is given by

$$\hat{S}_x^\alpha(f) = \frac{1}{2T_s} [Q(f+f_c+\alpha/2)Q(f+f_c-\alpha/2) + Q(f-f_c+\alpha/2)Q(f-f_c-\alpha/2)]e^{-i2\pi\alpha t_0} \quad (\text{A4a})$$

for $\alpha = k/T_s$ for all integers k , and $\hat{S}_x^\alpha(f) \equiv 0$ for all other values of α .

Equation (A4a) can be manipulated into the factored form

$$\hat{S}_x^\alpha(f) = [K(f+\alpha/2)K(\alpha/2-f) + L(f+\alpha/2)L(\alpha/2-f)]e^{-i2\pi\alpha t_0} \quad (\text{A4b})$$

where $K(f)$ is given by (A2c) (with ϕ_0 replaced with an arbitrary phase) and

$$L(f) \triangleq iQ(f+f_c)e^{-i(2\pi f_c t_0 + \phi_0)} - iQ(f-f_c)e^{+i(2\pi f_c t_0 + \phi_0)} \quad (\text{A4c})$$

in which ϕ_0 can be replaced with an arbitrary phase.

SQPSK: An SQPSK signal can be modeled the same as QPSK, except for the modification

$$c(t) = \sum_{n=-\infty}^{\infty} c_n q(t-t_0-T_s/2-nT_s) \quad (\text{A5})$$

($s(t)$ remains unmodified). It can be shown [3] that the spectral correlation function is given by (A4) for $\alpha = k/T_s$ for all even integers k , and it is given by

$$\begin{aligned} \hat{S}_x^\alpha(f) = \frac{1}{2T_s} [Q(f+f_c+\alpha/2) \\ \cdot Q(f+f_c-\alpha/2)e^{-i(2\pi[\alpha+2f_c]t_0+2\phi_0)} \\ + Q(f-f_c+\alpha/2) \\ \cdot Q(f-f_c-\alpha/2)e^{-i(2\pi[\alpha-2f_c]t_0-2\phi_0)}] \end{aligned} \quad (\text{A6})$$

for $\alpha = \pm 2f_c + k/T_s$ for all odd integers k , and $\hat{S}_x^\alpha(f) \equiv 0$ for all other values of α .

MSK: An MSK signal can be modeled the same as an SQPSK signal except that the rectangle pulse $q(t)$ is replaced with the positive half cycle of a cosine wave with period $2T_s$. It can be shown [3] that the spectral correlation function is the same as it is for SQPSK, except that $Q(f)$ is replaced with

$P(f)$ given by

$$P(f) = \frac{1}{4} [Q(f+1/2T_s) + Q(f-1/2T_s)]. \quad (\text{A7})$$

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