Subspace Signal Processing and Empirical Nonstationary FOT Expectation Operators

Given any finite (K) dimensional linear vector space spanned by any set of K linearly independent (not necessarily orthogonal or normal) basis functions, $\left\{\varphi_k(t)\right\}_1^K$ on the time interval $\left[0,T\right]$, with corresponding reciprocal basis $\left\{\theta_k(t)\right\}_1^N$ for which

$$\int_0^T \varphi_k(t)\theta_j^*(t)dt = \delta_{jk} \tag{1}$$

the linear operator with kernel (impulse response function) $h(t,u)$ given by

$$h(t,v) = \sum_{k=1}^{N} \varphi_k(t)\theta_k^*(v) \tag{2}$$

which projects onto the linear space, can be interpreted as an expectation operator for a time series that extracts the component of $x(t)$ or of any function $g(\cdot)$ of $\{x(t): t\in[0,T]\}$ that is contained in this linear space.

This expectation operator,

$$E^{\varphi}\left\{g(\mathbf{x}(t))\right\} = \int_0^T h(t,v)g(\mathbf{x}(v))dv\,, \tag{3}$$

where

$$g(x(t)) = g(x(t+t_1), x(t+t_2), ..., x(t+t_n))\,, \tag{4}$$

for any finite positive integer n, and it satisfies the Fundamental Theorem of Expectation,

$$E^{\varphi}\left\{g(\mathbf{x}(t))\right\} = \int g(\mathbf{x})\mathrm{d}F^{\varphi}(\mathbf{x};t) \tag{5}$$

where $F^{\varphi}(\mathbf{x})$ is the nonstationary cumulative distribution function defined by

$$\begin{aligned} F^{\varphi}(\mathbf{x};t) &\triangleq E^{\varphi}\left\{\prod_{j=1}^{n} u[x_j - x(t+t_j)]\right\} \\ &= \int_0^T h(t,v)\prod_{j=1}^{n} u[x_j - x(v+t_j)]dv \end{aligned}\,, \tag{6}$$

where $u(\cdot)$ is the unit step function. Consequently, once the projection operator has been applied to the indicator function for all values of $\mathbf{x}$ in the domain of the cartesian product of the n shifted versions $\left\{x(t+t_j)\right\}_1^n$ of $x(t)$, then the expected value of any function $g(\mathbf{x}(t))$ can be calculated using (5) for any well-behaved g.

It follows that an empirically derived nonstationary expectation operator of this type can be created for any time series on any finite time interval. However, the distribution defined by (6) is not a probability distribution unless the projection (3) is equivalent to a simple average, in which case 1) the 0-1 property of the indicator function guarantees that the distribution will take on values only in the interval [0, 1], and 2) the cumulative property of the continuous set of indicators indexed by $\mathbf{x}$ will ensure that the distribution is monotonically non-decreasing in $\mathbf{x}$. In this case the expectation operator is a probabilistic expectation operator. Otherwise, it is simply a component extraction operator, and the component it extracts from its argument is the portion of the argument that is fully contained in the subspace defined by the range of the projection operator.

Subspace methods of signal processing project data onto subspaces typically defined by the dominant eigenvectors of the data correlation or covariance matrix. So, apparently, an empirical ensemble or a stochastic process model is required for determination of the subspace onto which signals are to be projected. The projection is done using what is called the Karhunen-Loeve Transform. It is conceivable that useful subspaces can be specified for some applications without access to an ensemble of sample paths of the signal of interest or without a stochastic process model. For example, in some image processing algorithms, the vertically indexed set of horizontal image lines is treated as an ensemble of finite segments corresponding to horizontal lines. Yet the correlation in the vertical direction is not necessarily equal to that in the horizontal direction, though this can be corrected for by time scaling.

The time-average required to calculate cumulative probability distributions for continuous-time data can be continuous, in which case the derived CDF can be made stationary (time-invariant), as shown on page 3.5.3, or it can be discrete with a fixed period, in which case the derived CDF can be made cyclostationary (periodic), as shown on page 3.5.3. Also, the cyclostationary CDFs for multiple incommensurate periods can be combined to construct poly-cyclostationary (poly-periodic) CDFs.

The most generic form of the above *Fundamental Theorem of Non-Probabilistic Expectation* can be stated as follows: The image of a linear operator on "any" (nonlinear) transformation g (possibly vector-valued) of a time series $x(t)$ can be determined from the continuum of images of the linear operator on the continuum of indicator functions used to define a cumulative distribution function for $x(t)$. Then the image is just the integral of the transformation (as a function of a real variable, possibly vector valued) w.r.t. the CDF, as in (5). The result of this theorem is derived in the Appendix.

Appendix: Derivation of Fundamental Theorem of Expectation

To keep the notation simple, this derivation of the Fundamental Theorem of Expectation is carried out for a univariate distribution function $F_{x(t)}(x)$. And to keep the result as general as possible, we carry out the derivation for an arbitrary orthogonal projection operator, L. Consequently, "expectation" here means extraction of a component of a nonlinear or linear transformation $g[x(t)]$ of a time series $x(t)$.

We begin with the expression for the expectation of the transformed time series as a projection by a linear operator L and show that it can be re-expressed in terms of the cumulative distribution function. This orthogonal projection could be the continuous time average operation (constant component extractor), the discrete time average operation (the periodic component extractor), or any finite-dimensional subspace component extractor.

$$
\begin{aligned}
E\left\{g[\{x(t):t\in T\subseteq R\}\right\} &= L\left\{g[\{x(t):t\in T\subseteq R\}\right\} = L\left\{g[x(t)]\right\} \\
&= L\left\{\int_{-\infty}^{\infty} g(y)\delta\left(y-x(t)\right)\mathrm{d}y\right\} \\
&= \int_{-\infty}^{\infty} g(y)L\left\{\delta\left(y-x(t)\right)\right\}\mathrm{d}y \\
&= \int_{-\infty}^{\infty} g(y)L\left\{\frac{\mathrm{d}}{\mathrm{d}y}u\left(y-x(t)\right)\right\}\mathrm{d}y \\
&= \int_{-\infty}^{\infty} g(y)\frac{\mathrm{d}}{\mathrm{d}y}\left\{Lu\left(y-x(t)\right)\right\}\mathrm{d}y \\
&= \int_{-\infty}^{\infty} g(y)\frac{\mathrm{d}}{\mathrm{d}y}F_{x(t)}(y)\mathrm{d}y \\
&= \int_{-\infty}^{\infty} g(y)\mathrm{d}F_{x(t)}(y)
\end{aligned}
\quad , \qquad \text{(A.1)}
$$

where δ is the Dirac delta function,

$$F_{x(t)}(y) \triangleq L\{u(y-x(t)\} \qquad \text{(A.2)}$$

is the cumulative distribution function, and u is the indicator function which is zero for negative arguments and 1 for positive arguments.

For example, for a finite-dimensional subspace projector on a finite interval,

$$L\{u(x-x(t)\} = \int_0^T h(t,v)u(x-x(v))\mathrm{d}v \qquad \text{(A.3)}$$

where the kernel h is specified by (2).

As another example, for the constant component extractor, we have

$$L\{u(y-x(t)\} = \underset{T\to\infty}{Lim}\frac{1}{T}\int_{-T/2}^{T/2} u(y-x(t+v))\mathrm{d}v \quad . \qquad \text{(A.4)}$$