COMPLETING THE FAMILY OF NONSTOCHASTIC THEORIES OF TIME SERIES

Preliminary Definitions --

Def: *PURELY EMPIRICAL (THEORY/METHOD)*

- Excludes ensembles—particularly infinite ensembles—of outcomes of hypothetical experiments

- Excludes mathematical limits as some parameter, such as averaging time, approaches infinity.

- Excludes all quantities (e.g., Expected Values) that are not identifiable as, or cannot be calculated/computed from, recorded *physical* measurements or observations, including representations stored in media, e.g., computer memory.

- Consequence: As applied to time-series analysis, the mathematical descriptions of calculations are finite for discrete time and involve no limits for continuous time, other than that required in the definition of the Riemann integral over a finite interval.

Def: *STATISTICAL SPECTRUM*

- A *Statistical Spectrum* is an *empirically averaged* spectrum
- A *Probabilistic Spectrum* (e.g., the standard Power Spectral Density of a stochastic process) is a *mathematical quantity*
- Example of misuse: The *Statistical* Theory of Communication is mostly probabilistic, not statistical.

HIERARCHY OF NONSTOCHASTIC THEORIES OF TIME SERIES

<u>In order of *Ease of Mathematization*</u>

(starting with the lowest level of mathematical sophistication required, e.g., for proving existence of key quantities)

1) The ***Purely Empirical*** **NON-probabilistic theory**
   --<u>finite</u> time

   for <u>approximately</u> S, CS, and PolyCS (PCS) time series models (introduced in my 1987 book on statistical spectral analysis)

2) **NEW**: The ***Purely Empirical*** **FOT-probabilistic theory**
   --<u>finite</u> time

   for <u>exactly</u> S, CS, and PCS time series models (formally introduced here, 2021)

3) The ***Non-stochastic*** **FOT-probabilistic theory**
   --<u>infinite</u> time

   for <u>exactly</u> S, CS, PCS, and ACS (Almost-CS) time series (introduced in my 1987 book on statistical spectral analysis)

THE NON-STOCHASTIC FOT-PROBABILISTIC THEORY OF STATISTICAL SPECTRAL ANALYSIS IS NOT *PURELY EMPIRICAL*

- Existing theory for stationary and cyclostationary time series is not *purely empirical* because the key quantities in the theory are based on infinite limits of time averages and evaluation of these limits requires an analytical model of a time series, such as a mathematical formula, not just empirical measurements *represented by* mathematical symbols.

- The required property of *joint relative measurability* in the 2006 Leśkow -Napolitano theory cannot be verified empirically, because it requires analytical calculation based on an analytical model of an infinitely long time series.

- Strictly speaking, no times series can be said to be "at hand" if it is infinitely long. "At hand" is a term we have used since my 1987 SSA book to refer to a single time series as distinguished from a hypothetical ensemble of time series. But this term is used loosely when applied to infinitely long time series.

- Similarly, Almost-CS time series cannot be distinguished from Poly-CS time series in a *purely empirical* theory (*Poly-CS* means exhibits at most a *finite* number of harmonically unrelated cycle frequencies)

A *PURELY EMPIRICAL* NON-STOCHASTIC THEORY OF STATISTICAL SPECTRAL ANALYSIS EXISTS

- The motivation for using infinitely long time-averages is that it enables exact quantification (analogous to expected values), not just approximation, and it does not limit time separations in joint moments to finite values or limit spectral resolution of spectral moments to non-zero frequency intervals.

- But all averages in a *purely empirical* theory must be based on finite-time averages. Finite-length time series can indeed be "at hand".

- My 1987 SSA book presents an analytical non-probabilistic theory of statistical (time-averaged) spectral analysis that approximately quantifies temporal and spectral resolution and reliability (repeatability over time) for finite-length time series. [see parts of Chapters 2, 3, 7, 11, and 2 slides down from here]

- But this theory does <u>not</u> use the concept of probability—the closest thing to it that is used is the calculated finite-time temporal coefficient of variation of time-dependent measurements, such as spectral density and spectral correlation density.

A *PURELY EMPIRICAL* FOT-PROBABILISTIC THEORY OF STATISTICAL SPECTRAL ANALYSIS

- When the concept of probability is introduced in my 1987 SSA book, it is done in terms of infinite limits of time averages and it, therefore, forfeits the empiricism of the finite-time non-probabilistic theory.

- Yet, we CAN come quite close to an equivalent purely empirical theory based on finite-time averages, as long as we accept time separations in joint moments that are limited to finite values and spectral resolution of spectral moments that are limited to finite (non-zero) values, and we accept approximate quantification of <u>some</u> of the otherwise-exact relationships involving signal processing operations.

A *PURELY EMPIRICAL* FOT-PROBABILISTIC THEORY OF STATISTICAL SPECTRAL ANALYSIS – Examples

- Example 1: We can show that a statistical spectrum is approximately normal for sufficiently large averaging time, but we cannot prove it is asymptotically exactly normal, because infinite limits are outside the scope of the calculations allowed by an empirical theory.

- Example 2: We can show that the difference between time-averaged and frequency-smoothed statistical spectral correlation measurements can be made small when the temporal/spectral resolution product is large, but we cannot prove it is asymptotically zero, because infinite limits are outside the scope . . . This approximate relationship can be summarized as follows:

$$S^{\alpha}_{1/\Delta f}(t,f)\otimes r_{\Delta t}(t) \cong S^{\alpha}_{\Delta t}(t,f)\otimes r_{\Delta f}(f) \quad \text{for } \Delta t\Delta f \gg 1$$

$$\text{Temporal Resolution } \cong \Delta t$$

$$\text{Spectral Resolution } \cong \Delta f$$

$$\text{Cycle Resolution} \cong 1/\Delta \mathrm{t}$$

$$\text{Coefficient of Variation} \cong \left(1/\Delta t\Delta f\right)\left|\rho^{\alpha}_{\Delta t,\Delta f}(t,f)\right|^{-2}$$

$$\left|\rho^{\alpha}_{\Delta t,\Delta f}(t,f)\right|^{2} = \frac{\left|S^{\alpha}_{\Delta t,\Delta f}(t,f)\right|^{2}}{S^{0}_{\Delta t,\Delta f}(t,f+\alpha/2)S^{0}_{\Delta t,\Delta f}(t,f-\alpha/2)}$$

The left member of the approximation above is a time-averaged cyclic periodogram and the right member is a frequency-smoothed cyclic periodogram, the functions $r_{\Delta t}(t)$ and $r_{\Delta f}(f)$ are distinct smoothing windows of approximate width Δt and Δf, and $\otimes$ denotes convolution. The quantities in the righthand side of the coefficient-of-variation formula can be either time-averaged or frequency-smoothed cyclic periodograms: the approximation accommodates both.

ELEMENTS OF THE *PURELY EMPIRICAL* FOT-PROBABILISTIC THEORY

- Definitions of finite-time FOT *Cumulative Distributions* for S, CS, and PCS time-series models in terms of a new definition of Synchronized Average for finite-length time series and the introduction of reciprocal basis functions

- *These FOT Cumulative Distributions can be computed from empirical data*

- *Fundamental Theorems of Averaging and Sine-Wave Component Extraction* are the same as those for infinitely long time-series models

CONCLUSION

- As explained above and mathematically proved in the article below, it has been shown that there exists an entirely empirical FOT probabilistic theory of stationary, cyclostationary and poly-cyclostationary times series.

- All quantities occurring in the theory can be calculated from physically measured/observed time series data on finite intervals

- This theory should appeal to practitioners who analyze and process empirical data.