A Century of Evolution of Modeling Cycles in Time-Series Data

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Abstract

A concise review of the long evolution of the study of cycles in time-series data is provided as a basis for explaining the relationship between the half century of work on cycles by William A. Gardner between 1972 and the 2024 and the classical work of mathematicians and scientists throughout the preceding century including especially Norbert Wiener (generalized harmonic analysis, optimum filtering, nonlinear system identification), and also D. Brennan (Fraction-of-Time Probability), Ronald Fisher (cumulants), Lars Hanson (generalized method of moments), E.M. Hofstetter (Fraction-of-Time Probability), Andrei Kolmogorov (stochastic processes), Karl Pearson (method of moments), Arthur Schuster (periodogram), Thorvald Thiele (cumulants), John Wishart (cumulants), Herman Wold (hidden periodicity and disturbed harmonics), and many others who contributed to the theory of stationary stochastic processes and various topics in statistical signal processing based on the stationary process model.

Introduction to Cycles

The following introduction to the topic of the present essay was written by Herman O. A. Wold in 1968, as the opening paragraph in his survey contribution to the topic *"*Cycles*"* in *the Encyclopedia of the Social Sciences* [1].

> *Cycles, waves, pulsations, rhythmic phenomena, regularity in return, periodicity—these notions reflect a broad category of natural, human, and social phenomena where cycles are the dominating feature. The daily and yearly cycles in sunlight, temperature, and other geophysical phenomena are among the simplest and most obvious instances. Regular periodicity provides a basis for prediction and for extracting other useful information about the observed phenomena. Nautical almanacs with their tidal forecasts are a typical example. Medical examples are pulse rate as an indicator of cardiovascular status and the electrocardiograph as a basis for analysis of the condition of the heart.*
>
> *The study of cyclic phenomena dates from prehistoric times, and so does the experience that the area has dangerous pitfalls. From the dawn of Chinese history comes the story that the astronomers Hi and Ho lost their heads because they failed to forecast a solar eclipse (perhaps*

2137 b.c.). In 1929, after some twelve years of promising existence, the Harvard Business Barometer (or Business Index) disappeared because it failed to predict the precipitous drop in the New York stock market.

The purpose of this brief essay is to put into perspective the breakthrough made in the mid-1980s in the modeling of and statistical inference based on time-series data exhibiting cyclic behavior. Up until this breakthrough, statistical models for cycles—as a complement to nonstatistical cycles modeled, for example, by differential equations—had been studied analytically using crude mathematical models for more than a century but had not moved beyond the following two models: 1) the sum of one or more periodic time series and a featureless (randomly fluctuating, erratic, unpredictable, stationary) times series, often referred to as *noise*, which sum is amenable to more than just temporally local prediction, and 2) the response of a linear time-invariant resonant dynamical systems, mathematically modeled as a convolution, driven by a featureless time series, which response is amenable to only local prediction, because the apparent cycles are not true cycles. In a hypothesis testing setting, the null hypothesis (the alternative to models 1) or 2)) is an unpredictable nonstationary time series that may appear from time to time to exhibit cyclicity but that, upon closer inspection, is found to exhibit no true cycles and no substantive predictability. However, model 2) can be considered to be included in the null hypothesis since the disturbed harmonics produced by this model do not represent true cycles, and predictability is relatively limited. For an illustrative discussion of the general problem of cycles from a historical perspective, the reader is referred to Appendices 1 – 3, which consist of excerpts from Wold's article, "Cycles" [1].

The first method that emerged for analysis of data according to model 1), at the turn of the 19th Century, is the periodogram (the squared magnitude of the Fourier transform of a finite-length times series of data, normalized by the length of the data segment) which was followed by a variety of what were termed *high-resolution* and *super-resolution* model fitting methods beginning around mid-20th Century. The periodogram was proven to be the set of sufficient statistics for Maximum Likelihood (ML) estimation of the period of a cycle due to a single sinewave in additive white Gaussian noise (AWGN) and the amplitude and phase of the Fourier component at the detected period are ML estimates of a sinusoid with that period. The complexity of the generalization to ML estimation for multiple sinusoids in AWGN, especially those with cycle periods that are not substantially different, led to a wide variety of alternative model fitting method, which are surveyed in [3, chap 9], where Gardner introduces the use of the FOT probability model to circumvent the unnecessary abstraction of the stochastic process model (cf. [4]) which dominated the literature on this topic essentially to the extent of complete exclusion of the FOT probability model once the stochastic process had been introduced. Data following model 2) were referred to as *disturbed harmonics* and were analyzed primarily by methods developed specifically for Autoregressive Models (AR) and AR-Moving Average (ARMA) models. These models were initially implicitly based on the FOT model (i.e., on time averages of lag products, not probabilistic expected values) but soon transitioned to the stochastic process model.

Review of Cyclostationarity

In 1985 and 1987, two analytical books by William A Gardner [2], [3] appeared, and introduced the first comprehensive theoretical investigations of two new classes of models which he termed 4) *cyclostationary time series* exhibiting a single periodicity and its generalization to 5) *almost-cyclostationary time series* exhibiting multiple incommensurate periodicities, that is, multiple incommensurate *periods of statistical cyclicity.* (For a finite number of incommensurate cycles, Gardner later introduced the more specific term *poly-cyclostationary*.) Book [2] introduced these models in terms of stochastic processes and briefly explained their duals defined in terms of time averages instead of expected values, and [3] maintained close ties to empirical data by developing a comprehensive theory based on times averages alone or, equivalently, *Fraction-of-Time* (FOT) probabilities. The term *statistical cyclicity* means that precise cycles appear only in averages performed on the data, not in the raw data itself, which may or may not exhibit imprecise cycles. For the stochastic process model, these averages are expected values of functions of the data, which can be approximated with averages over statistical samples from a population of data sets. For the alternative non-stochastic model, these averages are ideally infinitely long time averages of functions of the data, which can be approximated by finite-time averages. The two models are dual and, in addition, they are essentially equivalent for a very special subclass of stochastic processes that satisfy the *ergodic hypothesis* [2, chap 8].

The original models 1) and 2) were first described prior to the advent of the concept of a stochastic process and later were replaced with stochastic-process alternatives. The two new models 4) and 5), which generalize models 1) and 2), were first treated comprehensively almost simultaneously in both forms, stochastic and non-stochastic, in [2], and the non-stochastic alternative was greatly expanded on in [3], because of its parsimony and more direct relevance to most applications—those for which only a single time series of measurements is available instead of a set of multiple statistical samples of time series from a population which is the situation originally motivating the stochastic process model. There were a few isolated journal papers prior to (and cited in) [2], [3], [5] and which briefly treated what were called *periodically correlated stochastic processes*, but there had been no attempt to develop a comprehensive theory of these stochastic processes, and not even a mention of the alternative theory of non-stochastic models for non-population time series first proposed in [2],[3] (cf. [4],[6]-[8]) let alone non-stochastic models for periodically and almost periodically time varying higher-than 2nd order moments, cumulants, and probability density functions. There also were a few isolated papers on stochastic cyclostationarity in the Russian literature that are cited in [5]. The fundamental concept underlying (almost) cyclostationarity does not require the concept or mathematical model of a population of time series and a corresponding stochastic process. Rather (almost) cyclostationarity can be defined in terms of time-series models consisting of (almost) periodically time-varying FOT *probability* density functions defined independently of the probability space notion upon which the stochastic process is defined. The reader is referred to [7] for a discussion of the underlying measure theory foundation for FOT probability, and to [4], [8] for discussions of the key mathematical differences between FOT probability, which is constructed from a single time series, and Kolmogorov's abstract axiomatically defined probability theory, which is defined in terms of what is called a *probability space*. Periodically (and almost periodically) time varying moments and cumulants can be characterized in terms of FOT probability. The breadth of this class of models and the phenomena to which they apply dwarfs the earlier models of cycles of type 1) referred to above. In fact, the model 1) is the most elementary example of a cyclostationary time series—so elementary that it does not need the mathematical machinery of FOT probability to analyze.

More specifically, in the model of type 1) a true cycle corresponds to a periodic mean and, in the model of type 2), an apparent but not true cycle corresponds to damped oscillation of the autocorrelation function of the process. In cyclostationary (or almost cyclostationary) processes, any order moment or cumulant can be periodic (or almost periodic with multiple incommensurate periods). For example, a cyclostationary process or time series can have a constant (time-invariant) mean and constant variance, but a periodic covariance producing cycles in coherence time; or it can have 1^{st} and 2^{nd} order moments all of which are constant, but periodic higher-order moments or cumulants. In general, (almost) cyclostationary processes have (almost) periodic joint probability density functions.

Gardner's more general FOT probability model of cycles does not rely on a hypothetical deterministic model (a periodic function or a convolution) mixed with or driven by a featureless noise. Rather, it constructs the model from time averages of functions of the time series. This model can consist of FOT probability density functions, joint moments of multiple time samples with any time separations, corresponding joint cumulants, etc. Nevertheless, the FOT probability model can be derived from a mathematical model of deterministic dynamics driven by featureless noise, in term of the FOT model of such noise, which is typically chosen to be a series of statistically independent identically distributed (in the FOT probability sense) variables. Several examples are listed below:

1) A piecewise constant time series with transitions once every period and with constants given by a stationary sequence. This model has constant mean and variance, but periodic covariance. The stationary sequence can be, for example, a featureless noise with known FOT probability, such as independent identically distributed variables.
2) A product of a deterministic periodic sequence and a stationary sequence (e.g., featureless noise) with known FOT probability. If this stationary noise is white, so too is the cyclostationary time series.
3) A marked and filtered Poisson point process (e.g., a detected photon stream) with average rate of occurrence of points that is a deterministic periodic function. The stationary sequence of random marks on the pulse shapes produced by the filter can be, for example, a featureless noise or an information bearing signal, as in optical communications systems.
4) A resonant dynamic system, with periodically time-varying resonant frequency and/or damping factor, driven by featureless noise.

5) A pulse stream with random amplitudes (e.g., featureless noise) and periodically time varying pace and/or duration.

The books [2] and [3], and a lifetime of follow-on work by the Author, comprehensively reviewed in [6], substantially extends and generalizes: Herman Wold's and George Yule's work on hidden periodicities and disturbed harmonics [1], [6, p. 4.1]; Norbert Wiener's work on Generalized Harmonic Analysis of stationary time-series [9] (generalized to spectral correlation analysis of cyclostationary and almost cyclostationary time series [6, p. 2.1] and further generalized from 2^{nd}-order joint moments to higher order moments and joint probability densities [6, p. 2.1]); Wiener's minimum time-averaged-squared-error (MTASE) linear time-invariant filtering of stationary time series [10] (generalized to periodically and almost periodically time-variant linear filtering of cyclostationary time series [6, p.2.5.1]); the body of work by Wiener and his team of PhD students at M.I.T. (cf. Zadeh's integrative formulation [11] and Materra's comprehensive review [6, p. 11.7]) on nonlinear system identification (reformulated in terms of fraction-of-time- probability and generalized from time invariant to (almost) periodically time variant nonlinear systems [6, p. 2.5.3]); and the work of many on blind channel identification/equalization (generalized from stationary to cyclostationary channel inputs thereby enabling measurement of phase as well as magnitude of the transfer function [6, p. 2.5.3]). Gardner's work also creates a parsimonious alternative to Andrei Kolmogorov's theory of Stochastic Processes [12], based on Gardner's concept of (almost) periodic fraction-of-time probability [6, p. 3], which is based on Gardner's non-population alternative to relative frequency for cyclostationary time series (a generalization of Brennan's and Hofstetter's early work on stationary time series [13],[14]), which also introduces an entirely innovative meaning of cumulants of non-population cyclostationary and almost cyclostationary time series [6, p. 2.1], the original meaning having been introduced by Thorvald Thiele in the late 19^{th} Century (cf. [15]) and called *semi-invariants* and later termed *cumulants* by Ronald Fisher and John Wishart.

In addition, Gardner's work extended and generalized the work of many on periodogram-based methods, following Arthur Schuster's introduction of the periodogram [16], for power spectral density estimation; that is, generalized to cyclic-periodogram-based methods for spectral correlation density estimation [2, p. 331], [3, p. 385].

Most recently, Gardner has furthered his earlier work on cyclostationarity by introducing a radically new Method of Moments (MoM) for model parameter estimation [17], [6, p. 11.4] as a competitive alternative to Karl Pearson's classic MoM (cf. [18]) from the turn of the 19^{th} Century (1894) and Lars Hansen's 1982 Generalized MoM [19].

The only other comprehensive treatment of Gardner's theory of cyclostationarity appeared over 3 decades after publication of his two books in an unusually scholarly and encyclopedic treatment in a 2019 book by Antonio Napolitano [20], who cites Gardner's founding work over 580 times.

The treatment of statistical time-series analysis in Gardner's two mid-1980s books is the first to argue at length that cyclostationarity modeling of time series data was missing from the preceding century of work on hidden periodicities, and a half century of work on disturbed harmonics; and that, from the mid-20th Century on, there is no apparent reason for this shortcoming in the development of time-series models and analysis other than the convenience of the availability of a mathematical theory of stationary stochastic processes and the fact that a seemingly harmless technique promoted by Blackman and Tukey in 1958 [2, page 357] can be used to render stationary a stochastic process otherwise exhibiting what became known as cyclostationarity—a property that had been intentionally avoided following Kolmogorov's introduction three decades earlier of stochastic processes. This "harmless" technique, called *phase randomization* [21], can indeed be quite harmful in terms of yielding higher-than-minimum-Bayes-risk statistical inferences based on the time series and its stationarized model [2], [3], and in terms of the masking of key properties, such as *spectral correlation*, the separability of spectral correlation among additive mixtures of time-series—often referred to as *signals*—and the separability of such signals themselves, and more general insight into statistical inference involving cycles [22]- [24].

In Wold's 1968 encyclopedia article [1], it is acknowledged that interest in the study of cyclicity waned following the transition from classical time-series analysis to the stationary stochastic process framework. This was an unfortunate setback in time series analysis of cycles that Gardner attributes to what Professor James Massey [6, p. 9.1] referred to as "the stochastic process bandwagon" in his review of the book [3] (cf. [6, p. 4]).

Almost thirty years after writing the treatise [3] on *Regular Cyclostationarity* (including *Regular Almost Cyclostationarity*), Gardner gave consideration to the alternative class of cyclostationarity complementary to regular cyclostationarity, which he termed *Irregular Cyclostationarity* (and *irregular almost cyclostationarity*). In 2015, the original unpublished version of the 2018 publication [25], as well as this 2018 article itself, revealed how to extend the cyclostationarity paradigm from regular to irregular cyclostationary times series. Irregular cyclostationarity is predominant in scientific data of natural origin in contrast to engineering data where cyclostationarity is often "manufactured" and intentionally made regular. Examples arise in communication systems design and analysis where the cyclicity in otherwise stationary data is intentionally introduced at the transmitter so that it can be used to advantage at the receiver for extracting the information content in the transmitted signal. Other examples arise in rotating machine monitoring and fault diagnosis where cyclicity is unavoidably introduced by motions of machine components such as rotating crankshafts, reciprocating pistons, and revolving bearings in internal combustion engines and electric motors and generators, including hydro-electric and wind turbines. This field of application of Gardner's theory was first proposed by Gardner in [3] and has since become a major field of study based on his theory.

Irregular cyclicity is specifically defined in [25] to be regular cyclicity after it has been subjected to time warping. This excludes other forms of departure from exact cyclicity that cannot be so modeled (e.g., pace-irregular pulsed time series described in [25], which arise in rotating machinery with time varying but non-periodic rpm. Irregular cyclicity is the more tractable departure of the two because it is amenable to mathematical modeling in terms of regular cyclostationarity and because irregular cyclostationary time series can be exactly or approximately converted to regular cyclostationarity. This work applies also to irregular almost cyclostationarity provided that all cycles are subjected to the same time warping. The model for irregular cyclostationarity generalizes the concept of a cycle which, by definition, is an exact periodicity to a special type of irregular cycle which is a time-warped periodicity. This significantly broadens the type of phenomena that can be advantageously modeled and predicted.

Not available in the open academic literature is decades of Gardner's and his research team's work applying his theory of cyclostationarity to the development of signal processing algorithms for signals intelligence for purposes of national security (cf. [26], [6, p. 12]). This work, reported in numerous treatises prepared for the government, revolutionized this field of study resulting in significant improvements in signals intelligence capability [6, quotation by Nelson Blackman on page 9.1]. Gardner's theory is also a cornerstone of today's work on spectrum sensing and management by spectral correlation analysis (cyclic spectrum analysis) as well as power spectral density estimation for cognitive radio systems [6, p. 11.1], [27].

Most recently, in 2024, Gardner, in collaboration with Napolitano, applied the FOT-probability model—as an alternative to the stochastic process model—to revisit the pros and cons of a spectrum estimation technique known as the *multi-taper method* (MTM) originally introduced by D. J. Thomson in 1982 [28], in comparison with classical methods (CMs) based on time averaging and/or frequency smoothing periodograms. The results of this work contradict the literature on this subject where superiority of the MTM over CMs is claimed [27]. This work illustrates the benefits achievable through simplified conceptualization by replacing the unnecessarily abstract stochastic process model with Gardner's parsimonious FOT probability model.

Conclusion

Many fields of application of the theory and methodology of (almost) cyclostationarity, other than those mentioned above, are listed in [6, pages 1, 4] and [20, chaps 7 (sec. 6), 9,10]. A concise list of Gardner's specific mathematical contributions to the theory and methodology based on his FOT-probability approach to non-population times series exhibiting cyclostationarity is provided in Appendix 4, which also addresses applications. Other unifying contributions to time series analysis made by Gardner also are outlined in Appendix 4. A detailed list of applications of cyclostationarity to a variety of fields of science and engineering is presented in Appendix 5.

The concise review provided by this essay illustrates Gardner's unusual approach to furthering our understanding of theory and methodology for statistical time series analysis. To quote the late Enders A. Robinson, past Professor of Geophysics at Columbia University, past Member of the National Academy of Engineering, and highest honored scientist in the field of geophysics, borrowing from his letter of reference to a Department Chairperson at University of California, Davis, on behalf of Professor Gardner [6, p. 9.1]:

> *From time to time it is good to look back and see in perspective the work of those people who have made a difference in the engineering profession. One of the important members of this group is William A. Gardner.*
>
> *Professor Gardner has the ability to impart a fresh approach to many difficult problems. William is one of those few people who can effectively do both the analytic and the practical work required for the introduction and acceptance of a new engineering method. His general approach is to go back to the basic foundations and lay a new framework. This gives him a way to circumvent many of the stumbling blocks confronted by other workers . . .*
>
> *I am particularly impressed by the fundamental work in spectral analysis done by Professor Gardner. Whereas most theoretical developments make use of ensemble averages, he has gone back and reformulated the whole problem in terms of time-averages. In so doing he has discovered many avenues of approach which were either not known or neglected in the past. In this way his work more resembles some of the outstanding mathematicians and engineers of the past. This approach took some courage, because generally people tend to assume that the basic work has been done, and that no new results can come from re-examining avenues that had been tried in the past and then dropped. William's success in the approach shows the strength of his engineering insight. He has been able to solve problems that others have left as being too difficult. It is this quality that he so well imparts to his students, who have gone forth and solved important and far-reaching problems in their own right.*

To provide a more concrete perspective on the substantial work on cycles that preceded the breakthrough in the mid-1980s, the reader is referred to [1]; also, a concise summary of the treatise [1] is provided in Appendices 1 - 3.

This essay is concluded here with a coarse timeline of the progression of recorded thought about cycles and corresponding data models from basic interest in cycles to the most sophisticated mathematical models yet to be devised:

- 2000 BC: Interest in the General Notion of Cycles (see excerpt from Wold [1] in the first paragraph of the present essay)

- 1700s AD: Hidden periodicities

  (Euler, Lagrange; see [3, p. 2])
- 1927: Disturbed Harmonics

  (Yule; see [3, p. 14])
- 1975 – 1978: Precursor to Regular (Almost) Cyclostationarity

  (Gardner; see [5], [21])
- 1985 – 1987: Regular (Almost) Cyclostationarity

  (first in-depth treatises: Gardner; see [2, Chap 13], [3, Part II)
- 2015 – 2018: Irregular Cyclostationarity [Gardner; see [25] and, for follow on work, see Napolitano, [29])

APPENDIX 1. H.O.A. Wold on Cycle

The following excerpt from Wold's encyclopedia article [1] on cycles provides an interesting discussion of the nature of the problem of detecting cycles and predicting the future. Several figures and equations that are present in [1] have been omitted from this excerpt without significantly detracting from the value of the discussion for the purposes of the present article. Similarly, in a few places, words (not italicized) have been inserted to facilitate comprehension where figures or equations have been omitted, as shown with a strikethrough line.

> *Cyclic phenomena are recorded in terms of time series. A key aspect of cycles is the degree of predictability they give to the time series generated. Three basic situations should be distinguished:*
>
> *(a) The cycles are fixed, so that the series is predictable over the indefinite future.*
>
> *(b) The cycles are partly random, so that the series is predictable only over a limited future.*
>
> *(c) The cycles are spurious—that is, there are no real cycles—and the series is not predictable.*
>
> *For the purposes of this article the term "cycle" is used in a somewhat broader sense than the strict cyclic periodicity of case (a).*

Limited and unlimited predictability

The fundamental difference between situations (a) and (b) can be illustrated by two simple cases.

The scheme of "hidden periodicities." *Suppose that an observed time series is generated by two components. The first is strictly periodic, with period length p, so that its value at time t + p is equal to its value at time t. The second component, superimposed upon the first, is a sequence of random (independent, identically distributed) elements. Thus, each term of the observed series can be represented as the sum of a periodic term and a random one.*

Tidal water is a cyclic phenomenon where this model applies quite well ~~(see Figure 1)~~. Here the observed series is the measured water level at Dover, the strictly periodic component represents the lunar cycle, 12 hours and 50 minutes in length (two maxima in one lunar day), and the random elements are the irregular deviations caused by storms, random variations in air pressure, earthquakes, etc.

The periodic component provides a prediction—

Hypothetical data. *An unbiased predicted value for a future time with expectation equal to that future value of the periodic component, and with prediction error equal to the random element. The difficulty is that the periodic component is not known and must be estimated empirically. A simple and obvious method is that of Buys Ballot's table; each point on the periodic component is estimated by the average of several points on the observed series, separated in time by the length of the period, p, where p either is known or is assessed by trial and error. The larger is the residual as compared to the cyclic component, the longer is the series needed to estimate with confidence the cyclic component.*

The approach of hidden periodicities may be extended, with two or more periodic components being considered. Tidal water again provides a typical illustration. In addition to the dominating lunar component, a closer fit to the data is obtained by considering a solar component with period 183 days.

In view of its simplicity and its many important applications, it is only natural that the approach involving strictly periodic components is of long standing. A distinction must be made, however, between formal representation of a series (which is always possible), on the one hand, and prediction, on the other. Under general conditions, any series, even a completely random one, can be represented by a sum of periodic components plus a residual, and if the number of *periodic components is increased indefinitely, the residual can be made as small as desired. In particular, if each of the periodic components is a sine or a cosine curve (a sinusoid), then the representation of the observed series is called a spectral representation. Such a representation, it is well to note, may be of only limited use for prediction outside the observed range, because if the observed range is widened, the terms of the representation may change appreciably. In the extreme case when the observations are all stochastically independent, the spectral representation of the series is an infinite sum of sinusoids; in this case neither the spectral representation nor alternative forecasting devices provide any predictive information.*

***Irregular cycles**. Until rather recently (about 1930), the analysis of oscillatory time series was almost equivalent to the assessment of periodicities. For a long time, however, it had been clear that important phenomena existed that refused to adhere to the forecasts based on the scheme of hidden periodicities. The most obvious and challenging of these was the sequence of some twenty business cycles, each of duration five to ten years, between 1800 and 1914. Phenomena with irregular cycles require radically different methods of analysis.*

***The scheme of "disturbed periodicity."** The breakthrough in the area of limited predictability came with Yule's model (1927) for the irregular 11-year cycle of sunspot intensity ~~(see Figure 2)~~. Yule interpreted the sunspot cycle as similar to the movement of a damped pendulum that is kept in motion by an unending stream of random shocks. [See the biography of Yule.]*

The sharp contrast between the scheme of hidden periodicities and the scheme of disturbed periodicity can now be seen. In the hidden periodicities model the random elements are superimposed upon the cyclic component(s) without affecting or disturbing their strict periodicity. In Yule's model the series may be regarded as generated by the random elements, and there is no room for strict periodicity. (Of course, the two types can be combined, as will be seen.)

The deep difference between the two types of model is reflected in their forecasting properties ~~(see Figure 3)~~. The time scales for the two forecasts have here been adjusted so as to give the same period. In the hidden-periodicities model the forecast over the future time span has the form of an undamped sinusoid, thus permitting an effective forecast over indefinitely long spans when the model is correct. In Yule's model the forecast is a damped sinusoid, which provides effective information over limited spans, but beyond that it gives only the trivial forecast that the value of the series is expected to equal the unconditional over-all mean of the series.

***Generalizations**. The distinction between limited and unlimited predictability of an observed times series goes to the core of the probability structure of the series.*

In the modern development of time series analysis on the basis of the theory of stochastic processes, the notions of predictability are brought to full significance. It can be shown that the series y_t under very general conditions allows a unique representation, known as predictive decomposition, where (a) the two components are uncorrelated, (b) one component *Φ_t is deterministic and* the other *Ψ_t is nondeterministic, and (c) the nondeterministic component allows a representation of the Yule type. In Yule's model no Φ_t component is present. In the hidden-periodicities model Φ_t, is a sum of sinusoids, while Ψ_t is the random Φ_t residual. Generally, however, Φ_t although deterministic in the prediction sense, is random.*

The statistical treatment of mixed models like this ~~*(1)*~~ *involves a variety of important and challenging problems. Speaking broadly, the valid assessment of the structure requires observations that extend over a substantial number of cycles, and even then the task is difficult. A basic problem is to test for and estimate a periodic component on the supplementary hypothesis that the ensuing residual allows a nondeterministic representation, or, more generally, to perform a simultaneous estimation of the two components. A general method for dealing with these problems has been provided by Whittle (1954); for a related approach, see Allais (1962).*

Other problems with a background in this decomposition occur in the analysis of seasonal variation [See Time series, article on Seasonal adjustment].

Other stochastic models. Since a tendency to cyclic variation is a conspicuous feature of many phenomena, stochastic models for their analysis have used a variety of mechanisms for generating apparent or genuine cyclicity. Brief reference will be made to the dynamic models for (a) predator-prey populations and (b) epidemic diseases. In both cases the pioneering approaches were deterministic, the models having the form of differential equation systems. The stochastic models developed at a later stage are more general, and they cover features of irregularity that cannot be explained by deterministic methods. What is of special interest in the present context is that the cycles produced in the simplest deterministic models are strictly periodic, whereas the stochastic models produce irregular cycles that allow prediction only over a limited future.

~~*Figure 4 refers to a*~~ Consider the *stochastic model given by M. S. Bartlett (1957) for the dynamic balance between the populations of a predator—for example, the lynx—and its prey—for example, the hare. The data of* ~~*the*~~ Bartlett's *graph are artificial, being constructed from the model by a Monte Carlo experiment. The classic models of A. J. Lotka and V. Volterra are deterministic, and the ensuing cycles take the form of sinusoids. The cyclic tendency is quite pronounced* ~~*in Figure 4*~~*, but at the same time the development is affected by random features.*

After three peaks in both populations, the prey remains at a rather low level that turns out to be critical for the predator, and the predator population dies out.

The peaks that have been observed *in ~~Figure 5~~* poliomyelitis data *mark the severe spells of poliomyelitis in Sweden from 1905 onward. The cyclic tendency is explained, on the one hand, by the contagious nature of the disease and, on the other, by the fact that slight infections provide immunity, so that after a nationwide epidemic it takes some time before a new group of susceptibles emerges. The foundations for a mathematical theory of the dynamics of epidemic diseases were laid by Kermack and McKendrick (1927), who used a deterministic approach in terms of differential equations. Their famous threshold theorem states that only if the infection rate, ρ, is above a certain critical value, ρ_o, will the disease flare up in epidemics. Bartlett (1957) and others have developed the theory in terms of stochastic models; a stochastic counterpart to the threshold theorem has been provided by Whittle (1955).*

Bartlett's predator-prey model provides an example of how a cyclic deterministic model may become evolutive (nonstationary) when stochasticized, while Whittle's epidemic model shows how an evolutive deterministic model may become stationary. Both of the stochastic models are completely nondeterministic; note that the predictive decomposition (1) extends to nonstationary processes.

The above examples have been selected so as to emphasize that there is no sharp demarcation between cycles with limited predictability and the spurious periodicity of phenomena ruled by randomness, where by pure chance the variation may take wavelike forms, but which provides no basis even for limited predictions. Thus, if a recurrent phenomenon has a low rate of incidence, say λ per year, and the incidences are mutually independent (perhaps a rare epidemic disease that has no aftereffect of immunity), the record of observations might evoke the idea that the recurrences have some degree of periodicity. It is true that in such cases there is an average period of length $1/\lambda$ between the recurrences, but the distance from one recurrence to the next is a random variable that cannot be forecast, since it is independent of past observations.

A related situation occurs in the summation of mutually independent variables. ~~Figure 6 shows a~~ A *case in point ~~as~~* is *observed in a Monte Carlo experiment with summation of independent variables (Wold 1965). The similarity between the three waves, each representing the consecutive additions of some 100,000 variables, is rather striking. Is it really due to pure chance? Or is the computer simulation of the "randomness" marred by some slip that has opened the door to a cyclic tendency in the ensuing sums? (For an amusing discussion of related cases, see Cole's "Biological Clock in the Unicorn" 1957.)*

~~*Figure 6 also gives, in the*~~ A *series of wholesale prices in Great Britain*~~*, an*~~ provides an *example of "Kondratieff waves"—the much discussed interpretation of economic phenomena as moving slowly up and down in spells of some fifty years. Do the waves embody genuine tendencies to long cycles, or are they of a spurious nature? The question is easy to pose but difficult or impossible to answer on the basis of available data. The argument that the "Kondratieff waves" are to a large extent parallel in the main industrialized countries carries little weight, in view of international economic connections.* ~~*The two graphs have been combined in Figure 6 in order to emphasize that*~~ *With regard to* observed *waves of long duration it is always difficult to sift the wheat of genuine cycles from the chaff of spurious periodicity. [See the biography of Kondratieff.]*

The bibliography for Wold's article [1] is included here in Appendix 3.

APPENDIX 2: Further Remarks on Cycles from H.O.A Wold

Cycles are of key relevance in the theory and application of time series analysis; their difficulty is clear from the fact that it is only recently that scientific tools appropriate for dealing with cycles and their problems have been developed. The fundamental distinction between the hidden-periodicity model, with its strict periodicity and unlimited predictability, and Yule's model, with its disturbed periodicity and limited predictability, could be brought to full significance only after 1933, by the powerful methods of the modern theory of stochastic processes.

On the basis of the FOT theory and methodology of interest in this article, as an alternative to the stochastic process, the veracity of the above phrase "*could be brought to full significance only after 1933, by the powerful methods of the modern theory of stochastic processes*" is questionable—the only randomness in the stochastic models discussed by Wold that cannot be incorporated in this parsimonious alternative are those which render the stochastic processes non-ergodic; namely, any time-invariant random parameters in the models discussed.

On the applied side, the difficulty of the problems has been revealed in significant shifts in the very way of viewing and posing the problems. Thus, up to the failure of the Harvard Business Barometer the analysis of business cycles was essentially a unirelational approach, the cycle being interpreted as generated by a leading series by way of a system of lagged relationships with other series. The pioneering works of Jan Tinbergen in the late 1930s broke away from the unirelational approach. The models of Tinbergen and his followers are multirelational, the business cycles being seen as the resultant of a complex system of economic relationships. [See Business cycles; Distributed lags.]

The term “cycle,” when used without further specification, primarily refers to periodicities in time series, and that is how the term is taken in this article. The notion of “life cycle” as the path from birth to death of living organisms is outside the scope of this presentation. So are the historical theories of Spengler and Toynbee that make a grandiose combination of time series and life cycle concepts, seeing human history as a succession of cultures that are born, flourish, and die. Even the shortest treatment of these broad issues would carry us far beyond the realm of time series analysis; this omission, however, must not be construed as a criticism. [For a discussion of these issues, see Periodization.]

***Cycles vs. innovations**. The history of human knowledge suggests that belief in cycles has been a stumbling block in the evolution of science. The philosophy of the cosmic cycle was part of Stoic and Epicurean philosophy: every occurrence is a recurrence; history repeats itself in cycles, cosmic cycles; all things, persons, and phenomena return exactly as before in cycle after cycle. What is it in this strange theory that is of such appeal that it should have been incorporated into the foundations of leading philosophical schools and should occur in less extreme forms again and again in philosophical thinking through the centuries, at least up to Herbert Spencer, although it later lost its vogue? Part of the answer seems to be that philosophy has had difficulties with the notion of innovation, having, as it were, a horror innovationum. If our philosophy leaves no room for innovations, we must conclude that every occurrence is a recurrence, and from there it is psychologically a short step to the cosmic cycle. This argument being a blind alley, the way out has led to the notions of innovation and limited predictability and to other key concepts in modern theories of cyclic phenomena. Thus, in Yule’s model ~~(Figure 2)~~ the random shocks are innovations that reduce the regularity of the sunspot cycles so as to make them predictable only over a limited future. More generally, in the predictive decomposition ~~(1)~~ the nondeterministic component is generated by random elements, innovations, and the component is therefore only of limited predictability. Here there is a close affinity to certain aspects of the general theory of knowledge. We note that prediction always has its cognitive basis in regularities observed in the past, cyclic or not, and that innovations set a ceiling to prediction by scientific methods. [See Time series, article on Advanced problems.]*

This article aims at a brief orientation to the portrayal of cycles as a broad topic in transition. Up to the 1930s the cyclical aspects of time series were dealt with by a variety of approaches, in which nonscientific and prescientific views were interspersed with the sound methods of some few forerunners and pioneers.

Regarding the following paragraph from Wold [1], it is suggested that readers consider the present article’s proposal that the interest in cycles being superseded as time-series analysis transitioned to the stochastic-process framework, as described below, is a result of the fact that the stationary process model, with the possible addition of sinewaves, is simply not appropriate for most cyclic phenomena; the supersession is not a result of cyclicity in data no longer being of interest. This proposal is supported by the huge growth in research on cyclicity in data since the advent of the theory and method of cyclostationarity a couple of decades later.

> *The mathematical foundations of probability theory as laid by Kolmogorov in 1933 gave rise to forceful developments in time series analysis and stochastic processes, bringing the problems about cycles within the reach of rigorous treatment. In the course of the transition, interest in cycles has been superseded by other aspects of time series analysis, notably prediction and hypothesis testing. For that reason, and also because cyclical features appear in time series of very different probability structures, it is only natural that cycles have not (or not as yet) been taken as a subject for a monograph.*

APPENDIX 3: H.O.A Wold's Bibliography

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APPENDIX 4: Gardner's Specific Mathematical Contributions

This appendix contains excerpts from [6, p.9.1], and the citations herein refer to the bibliography in [6, p.9.1],

Terminology Introduced by Gardner

1. Cycle Aliasing **[Bk2, p. 403, 528]**
2. Cycle Detector **[Bk1, p.352]**, **[Bk2, pp. 497-503]**, **[JP27]**
3. Cycle Frequency **[Bk1, p. 303]**, **[Bk2, p. 385]**
4. Cycle Leakage **[Bk2, p. 528]**

5. Cycle Resolution [Bk2, p. 388]
6. Cycle Spectrum [Bk1, p. 304], [Bk2, p. 392]
7. Cyclic Autocorrelation [Bk1, p. 303], [Bk2, p. 3]
8. Cyclic Expectation [Bk2, pp. 517-519]
9. Cyclic Correlogram [Bk1, p. 309], [Bk2, p. 386]
10. Cyclic Cumulative FOT Distributions and Densities [Bk2, pp. 511-515]
11. Cyclic FOT Probability [Bk2, pp. 511-515]
12. Cyclic Periodogram [Bk1, p. 309], [Bk2, p. 385]
13. Cyclic Polyspectrum [JP55]
14. Cyclic Spectral Density [Bk1, p.304], [Bk2, p. 559]
15. Cyclic Spectrum [Bk1, p. 304], [Bk2, p. 365]
16. Cyclic Temporal Cumulants [JP55]
17. Cyclic Temporal Moments [JP55]
18. Cyclic Wiener Filter [JP48], [Bk2, p. 482]
19. Cyclic Wiener Relation [JP36, p. 22], [Bk2, p. 390]
20. Cycloergodicity [Bk1, pp. 435-349], [JP11]
21. Poly-periodic Component Extraction Operator [JP34, pp. 282-284]
22. Poly-periodic Cumulative Probability Distribution [Bk1, p.348], [Bk2, pp. 512], [JP34, p. 283]
23. Pure n-th Order Cycle Frequency [JP55]
24. Purely Cyclostationary [Bk2, p. 392]
25. Sine-Wave Component Extraction Operator [Bk2, pp. 517-519], [JP34, pp. 280-284]
26. Spectral Autocoherence [JP15, p. 20], [Bk2, p. 366]
27. Spectral Correlation Density Function [Bk1, p. 304]
28. Spectral Cumulants [JP55]
29. Spectral Line Generation [Bk2, p. 359-369]
30. Spectral Moments [JP55]
31. Synchronized Average [Bk1, p. 311], [JP15, p. 17]

Theorems from Gardner's Theory of Cyclostationarity

CATEGORY 1: Probabilistic and Statistical Functions of Time and Frequency

Theorem 1: *Fundamental Theorem of Sine-Wave Component Extraction* – Original definition of the linear operator that extracts finite-strength additive sinewave components from any well behaved function of a persistent times series and proof that this operator is a nonstochastic expectation operator with respect to the generally almost period temporal (non-stochastic) cumulative probability distribution function, which also was originally defined by Gardner and proven to satisfy the defining properties of probability distribution functions in the case for which the set of sinewave frequencies comprises all harmonics of any set of incommensurate fundamental frequencies included. For one fundamental frequency, the distribution function provides a cyclostationary model for time series; for a finite set of multiple fundamental frequencies, it produces a poly-cyclostationary model, and for a countable infinity of fundamental frequencies, it produces an almost cyclostationary model.

Theorem 2: *Non-Stochastic Moment Expansion Theorem for Sine Waves Extraction* – Original theorem statement and proof that the set of sine waves to be extracted from any function of a time series that admits a generalized Volterra series representation can be expressed as a linear combination of the sine waves contained in each of all the moments of the time series or, equivalently, all the cumulants.

Theorem 3: Generalizations of the Fundamental Theorem of Sine Wave Component Extraction to extraction of subsets of sine waves and to Estimation on Finite-Time intervals – Completely analogous to Theorem 1, which applies to infinite intervals and complete sets of harmonically related sine waves. Because sine waves with arbitrary frequencies are not generally orthogonal to each other on finite intervals, this theorem deals with *estimation* instead of *extraction.*

Theorem 4: Sine-Wave-Extraction Derivation of the Non-Stochastic Temporal Cumulant Function – Original definition and derivation outside the framework of probability.

Theorem 5: Relation between non-stochastic Higher-Order Cyclic Moments and Cyclic Cumulants – Original derivation of the non-stochastic counterpart of the *Leonov/Shiryaev Relation* between stochastic moments and cumulants and its decomposition into the entirely new relation among cyclic moments and cyclic cumulants for a non-stochastic time series exhibiting cyclostationarity.

Theorem 6: Relation between Non-Stochastic Spectral Correlation and Cyclic Temporal Autocorrelation – Generalization of the Wiener Relation between non-stochastic average power spectral density function and temporal autocorrelation function: Original derivation.

Theorem 7: Relation between Non-Stochastic Temporal and Spectral Higher-Order Moments: Original derivation.

Theorem 8: Cyclic-Periodogram/Correlogram Relation and its Higher-Order Counterpart – Original definitions and derivations.

Theorem 9: Synchronized Averaging Identity for Non-Stochastic FOT-Probabilistic Functions – Original derivation.

CATEGORY 2: Transformation of Probabilistic and Statistical Functions by Signal Processing Operations

Theorem 10: Non-Stochastic Spectral-Correlation Input/Output Relations for Key Signal Processing Operations: 10.1, sampling & aliasing; 10.2, multiplication; 10.3, convolution & band limitation.

Theorem 11: Non-Stochastic Higher-Order Spectral-Moment and Spectral Cumulant Input/Output Relations for Key Signal Processing Operations: 11.1; 11.2; 11.3.

Theorem 12: Derivation of Signal Selectivity of Non-stochastic Cyclic Cumulants

CATEGORY 3: Optimum Statistical Inference

Theorem 13: Non-Stochastic Spectral Correlation Theory of Optimum Almost-periodically Time-Variant Filtering of Almost-Cyclostationary Signals: Generalization of the theory of non-causal Wiener filtering from stationary to almost cyclostationary time series. Includes the special cases of optimum poly-periodic and optimum periodic filtering of poly-cyclostationary and cyclostationary signals, respectively.

Theorem 14: Non-Stochastic Spectral Correlation Theory of Optimum Detection of Cyclostationary Signals: Maximum-SNR and Maximum-Likelihood spectral-line regenerators.

Theorem 15: Non-Stochastic Spectral Correlation Theory of N-th Order Nonlinear Synchronizers

Theorem 16: Non-Stochastic Spectral Moment and Cumulant Theory of Cyclostationary Signal Classification

Theorem 17: Non-Stochastic Theory of Almost-Periodically Time-Variant Linear System Identification

Theorem 18: Non-Stochastic Theory of Time-Invariant Volterra Nonlinear System Identification

Theorem 19: Non-Stochastic Theory of Periodically and Poly-Periodically Time-Variant Nonlinear Volterra System Identification

Notes and References for Theorems

- Re: Theorems 1 – 19 — Origin of the *Fraction-of-Time (FOT) Theory of ACS Time-Series*, which is dual to the Theory of ACS Stochastic Processes which reflects the *Gardner Isomorphism* which is the ACS counterpart of the *Wold Isomorphism* for stationary stochastic processes. See p. 62 in **[B2]** and Page 3.2 herein for CS; for ACS, see **[JP34]** and pp. 519-520 in **[Bk2]** and Page 3.2 herein.
- Re: Theorem 1 — Origin of the *Theorem of Temporal Expectation for ACS time-series based on the Sine-Wave-Component-Extraction Operator.* This is the FOT dual to the standard *Fundamental Theorem of Expectation* applied to ACS stochastic processes. See pp. 43-50, 137-138 in **[B2]** and pp. 517-519 in **[Bk2]**. The proof of the theorem given on pp. 11 -13 in **[Bk5]** is as elegant as possible, consisting of two simple steps, and this novel method of proof also extends to the classical Fundamental Theorem of Expectation for stochastic processes.
- Re: Theorem 2 — Origin of the Moment Expansion Theorem. This is formulated and proven on pp. 11 – 13 in **[Bk5]**, where the most elegant proof possible, consisting of two simple steps, is given.
- Re: Theorem 3 — Origin of the fact that the theory of cyclostationarity (including poly-cyclostationarity and almost-cyclostationarity), which is based on infinite limits of time average operations—non-empirical quantities—has a counterpart based on empirical finite-time averages. (See **Page 3.5** herein.)
- Re: Theorem 4 — Origin of the cumulant solution, in a non-probabilistic setting, to a problem concerning sine-wave generation by nonlinear transformation of persistent time series, see p. 146-149, 510 in **[B2]**. The solution provided on p. 3396 in **[JP55]** was proposed by Gardner and verified by C.M. Spooner. This solution was then characterized by Gardner and Spooner in terms of the cyclic cumulative distribution function and cyclic moments on pp. 3397-3399 in **[JP55]**.
- Re: Theorem 5 — Origin of the *Cyclic Moment/Cyclic Cumulant Relation* for times-series exhibiting cyclostationarity: This is a non-stochastic counterpart of the *Leonov/Shiryaev Relation* between moments and cumulants of a nonstationary stochastic process, pp. 147-148 in **[B2]**. This counterpart decomposes the relation for a stochastic processes into a set of relations among the cyclic components of the moments and cumulants for the cases of processes and time series exhibiting cyclostationarity **[JP55]**.
- Re: Theorem 6 — Origin of the *Relation* between the Cyclic Autocorrelation Function and the Spectral Correlation Function (originally dubbed *Cyclic Wiener Relation* by Gardner because it is an extension and generalization of the *Wiener Relation*, a term Gardner introduced to distinguish this relation from the *Wiener-Khinchin Relation* for stochastic processes). See page 390 in **[Bk2]**, and pp. 10, 20, 56, 57, 139 in **[B2]**.

- Re: Theorem 7 — Origin of the *Relation between Higher-Order Temporal and Spectral Moments* of time-series, pp. 138,139 in **[B2]**.This is the non-stochastic counterpart of what is called the *Shiryaev-Kolmogorov Relation* which is the higher-order generalization of the 2nd order *Wiener-Khinchin Relation,* see **[JP55]**. This relation is derived from the empirically-motivated definition of the spectral moment as the limit of the joint moment of finite-time Fourier transforms of a time series, rather than—as in **[B2]**–obtained by defining the spectral moment to be the Fourier transform of the cyclic temporal moment function.
- Re: Theorem 8— Origin of the *Cyclic Periodogram/Correlogram Relation* for time-series (see p. 57 in **[B2]** and pp. 385-386 in **[Bk2]**) and it's n^{th}-order counterpart (see p. 3419 in **[JP56]**), which is the finite-time counterpart of Theorem 7.
- Re: Theorem 9 — Origin of the *Synchronized Averaging Identity* for functions containing an additive almost period component. This identity decomposes that component into a sum of periodic components each derived directly from the time series and decomposes each periodic component into a sum of sinusoidal components each derived directly from the time series, pp. 485-486 in **[B2]**. See also pp. 362-365 and 511-515 in **[Bk2]** and pp. 332-334 in **[Bk3]**.
- Re: Theorem 10, 11 — Origin of the *Spectral Correlation Characteristics* of basic signal processing operations (time-sampling & aliasing, multiplication, convolution, and band limitation), pp. 82-109 in **[B2]** and **[JP15]**; and *Generalization to Higher-Order Moments/Cumulants*, **[JP56]**, pp. 133- 149 in **[B2]**, and Chap. 2 in **[Bk5]**.
- Re: Theorem 13 — Invention of *Cyclic Wiener Filtering Theory,* also called *FRESH Filtering*, and proof that Fractionally-Spaced-Equalizers are Cyclic Filters with Subsampled Outputs, pp. 330-333 in **[B2]**. See also pp.482-485 in **[Bk2]** and **[JP48]**. This seminal work gave rise to significant progress in multi-user receiver filter optimization and especially joint receiver/transmitter filter optimization, Article 1 in **[Bk5]**. It also provides the theoretical basis for separation of spectrally overlapping signals exhibiting cyclostationarity by exploitation of spectral redundancy.
- Re: Theorem 14 — Invention of the *Single-Cycle Detector* and proof that it is a *Maximum-Signal-to-Noise-Ratio sine-wave generator,* and original derivation of the decomposition of the maximum-likelihood detector for weak ACS signals into a coherent sum of Max-SNR Cycle Detectors, pp. 286-290 in **[B2]**. See also pp, 497-502 in **[Bk2]** and **[JP27]**, **[JP41]**, and **[JP49]**.
- Re: Theorem 15 — Origin of the central role that N^{th}-order spectral correlation plays in the operation of *N^{th}-Order Nonlinear Synchronizers for ACS signals*, pp. 333-335 in **[B2]**. See also Article 2 in **[Bk5]** and **[JP17]**.
- Re: Theorems 12, 16 — Original discovery of the signal selectivity property of cyclic temporal and spectral cumulants, and demonstration of utility for classification of spectrally overlapping signals and spectrum sensing for cognitive radio, pp. 322-324, 328-329 in **[B2]**. See also pp. 371-375 in **[Bk3]**, pp. 8-9, 65-66, 115 in **[Bk5]**, and pp.3399-3400 in **[JP55]**. This separability property

of cyclic correlations has also spawned an important new class of *Super-Resolution Direction Finding Algorithms*, the first of which were Cyclic MUSIC and Cyclic ESPRIT. Surveys provided in **[BkC1]** and Chap. 3 in **[Bk5]**.

- Re: Theorem 17 — Original discovery that *Blind Phase-Sensitive Channel Identification/Equalization* is made possible with only 2nd-order statistics by exploiting the cyclostationarity of channel-input signals, p.343 in **[B2]**. This discovery, first reported in **[JP39]**, did not include a particularly attractive algorithm to demonstrate this new capability, but it immediately gave rise to a flurry of contributions by other researchers to blind-adaptive channel equalization using only 2nd-order cyclic statistics. See Articles 4 and 5 in **[Bk5]**; surveys provided in **[BkC2]** and Chap. 3 in **[Bk5]**.
- Re: Theorems 11, 17-19 — Original discovery of *Input/Output-Corruption-Tolerant Linear System Identification Methods*, which are made possible by exploiting cyclostationarity of an input-signal component, pp. 335-337 in **[B2]** (see also **[JP31]**); and original extension and generalization of *Volterra Nonlinear System Identification* methods by exploiting cyclostationarity of the excitation, p. 344 in **[B2]**. Benefits for time-invariant nonlinear system identification are demonstrated mathematically in the originating paper where the methods are derived **[JP50]** and what are apparently the first-ever methods proposed for periodic and poly-periodic nonlinear system identification are demonstrated mathematically in the paper where they are derived **[JP59]**.

Applications of Theorems

- The above theorems gave rise to the discovery, for CS and ACS time series, of the fundamental *noise and interference tolerance properties in statistical inference* and the *Signal-Separability, Spectral Correlation Separability, and more generally Cyclic Temporal and Spectral Cumulant Separability* properties, and demonstration of applicability to design and analysis of signal processing methods and algorithms for communications, telemetry, and radar systems. This body of work has demonstrated that substantial improvements in system performance can be obtained in various signal processing applications involving multiuser communications and interference-limited environments, such as detection, estimation, and classification of signals, by exploiting cyclostationarity—that is, by recognizing and modeling signals as CS and ACS instead of using the classical stationary-process models. **Google Scholar** identifies more than 50 of Gardner's published research papers in peer-reviewed journals, and books, in which these achievements are reported, and identifies tens of thousands of research papers that cite this work.
- These theories and methods have provided the basis for the establishment of the core of a major part of RF signals intelligence algorithm development throughout government

laboratories & agencies and industrial government-contractors in the US and cooperating nations since the mid-1980s. Most of this work is not published in the open literature. (See SSPI Reports on **Page 6** and **Page 12** herein, and also see the Quotations from Nelson Blachman and Bart Rice below.)

- These theories and methods have provided the basis for new interference-tolerant signal processing techniques of signal-presence detection, parameter estimation, system identification, modulation classification, signal-source location and signal extraction; for example, these theories and methods have been adopted as the basis for spectrum sensing in crowded RF environments upon which the entire operation of cognitive radios relies. They have been used to develop a variety of substantive new families of algorithms for signal direction-of-arrival estimation, blind adaptive channel identification and equalization, blind adaptive spatial filtering, single-sensor cochannel signal separation, multi-user joint receiver/transmitter filter optimization, and nonlinear system identification. (See **Page 2.5** herein and the reference lists in **[JP64]** and in the various chapters and articles in **[Bk5]** and **[B2]**.)
- These theories and methods have also provided the basis for improved time-series analysis in a variety of other fields of science and engineering, such as Econometrics, Biology, Climatology, Acoustics and Mechanical Vibrations, and Electrical Circuits, Systems, and Control. See **Page 6** herein, and the pages in **[B2]** that are cited on Page 6. In fact, major advances in rotating machinery monitoring and early diagnosis of machine faults, such as gear and bearing degradation have been based on cyclostationarity exploitation, pp. 360-362 in **[B2]**.
- Development of the ad hoc concept of time de-warping into the basic theory of converting irregular cyclostationarity into regular cyclostationarity has served as a means for rendering the extensive and powerful tools of cyclostationary signal processing technology applicable to data exhibiting irregular cyclicity—when the rate of change of cycle frequencies is not too fast—which pervades essentially all fields of science. (See **Page 4.2** herein.)

Gardner's work also includes various unifications of statistical concepts and methods, and novel interpretations of fundamental entities including the likelihood sensitivity characterization of the Cramer Rao bound on variance of parameter estimators [JP7], and the establishment of the fact that conditional probabilities [JP4] and conditional expectations, pp. 427-428 in [Bk2] are orthogonal projections. These contributions include

1) Unification of bias, variance, spectral resolution, and spectral leakage properties through representations of all the common direct statistical spectral analysis methods as quadratic time invariant transformations fully characterized by their quadratic operator kernels [Bk2]
2) Unification of second-order statistical measures for signal classification [JP8]

3) Unification of numerous ad hoc *feature detectors* used for detecting signals with unintended receivers by characterizing all these features in terms of the statistical properties of cyclostationarity called Cyclic Spectral Densities and Higher-Order Spectral Moments/Cumulants [JP27]
4) The unifying concept, for blind adaptive filtering methods, of property restoral including modulus properties and spectral correlation properties [JP32]
5) A Unifying View of Spectral Coherence in Signal Processing [JP45] and On the Spectral Coherence of Nonstationary Processes [JP37]
6) Unification over all noise PDFs via the Structural Characterization of Locally Optimum Detectors in terms of Locally Optimum Estimators and Correlators [JP10]
7) Unification of Structurally Constrained Receivers for Detection and Estimation in terms of estimated posterior probabilities [JP4]
8) Unified characterization of modulation-rate distortion resulting from the most common analog circuit realizations of frequency modulators [JP1]
9) A unified non-population probabilistic theory for stationary and cyclostationary time-series analysis based entirely on time averages.

APPENDIX 5 List of Application Areas Using the Theory of Cyclostationarity

CITATIONS FOUND WITH A GOOGLE SCHOLAR SEARCH

Table 1 Nearly Distinct Application Areas

Taken from: Statistically inferred time warping: extending the cyclostationarity paradigm from regular to irregular statistical cyclicity in scientific data

	SEARCH TERMS[1]	# of HITS
1	"aeronautics OR astronautics OR navigation" AND "CS/CS"	3,190
2	"astronomy OR astrophysics" AND "CS/CS"	864
3	"atmosphere OR weather OR meteorology OR cyclone OR hurricane OR tornado" AND "CS/CS"	2,230
4	"cognitive radio" AND "CS/CS"	8,540

5	"comets OR asteroids" AND "CS/CS"	155
6	"cyclic MUSIC"	512
7	"direction finding" AND "CS/CS"	1,170
8	"electroencephalography OR cardiography" AND "CS/CS"	742
9	"global warming" AND "CS/CS"	369
10	"oceanography OR ocean OR maritime OR sea" AND "CS/CS"	3,060
11	"physiology" AND "CS/CS"	673
12	"planets OR moons" AND "CS/CS"	274
13	"pulsars" AND "CS/CS"	115
14	"radar OR sonar OR lidar" AND "CS/CS"	5,440
15	"rheology OR hydrology" AND "CS/CS"	639
16	"seismology OR earthquakes OR geophysics OR geology" AND "CS/CS"	1,090
17	"SETI OR extraterrestrial" AND "CS/CS"	83
18	autoregression AND "CS/CS"	2,040
19	bearings AND "CS/CS"	3,980
20	biology AND "CS/CS"	2,030
21	biometrics AND "CS/CS"	309
22	chemistry AND "CS/CS"	2,020
23	classification AND "CS/CS"	10,900
24	climatology AND "CS/CS"	811
25	communications AND "CS/CS"	21,200
26	cosmology AND "CS/CS"	172
27	ecology AND "CS/CS"	356

28	economics AND "CS/CS"	2,050
29	galaxies OR stars AND "CS/CS"	313
30	gears AND "CS/CS"	2,000
31	geolocation AND "CS/CS"	676
32	interception AND "CS/CS"	2,270
33	mechanical AND "CS/CS"	4,770
34	medical imaging OR scanning AND "CS/CS"	1,370
35	medicine AND "CS/CS"	2,990
36	modulation AND "CS/CS"	17,000
37	physics AND "CS/CS"	4,539
38	plasma AND "CS/CS"	542
39	quasars AND "CS/CS"	47
40	Sun AND "CS/CS"	4,320
41	UAVs AND "CS/CS"	238
42	universe AND "CS/CS"	209
43	vibration OR rotating machines AND "CS/CS"	3,240
44	walking AND "CS/CS"	990
45	wireless AND "CS/CS"	15,100
	TOTAL	135,628

1 "CS/CS" is an abbreviation for "cyclostationary OR cyclostationarity"

Table 2 Partially Redundant General Subjects[a]

From: [Statistically inferred time warping: extending the cyclostationarity paradigm from regular to irregular statistical cyclicity in scientific data]()

1	"almost "CS/CS""	8,840
2	"almost periodically correlated" AND "sequences OR processes"	352
3	"Cyclic Wiener Filtering" OR "FRESH filtering" OR "frequency-shift filtering"	676
4	"cyclostationary EOF" OR "cyclostationary empirical orthogonal functions" OR CSEOF	453
5	"exploiting "CS/CS""	11,900
6	"Gardner relation" OR "Cyclic Wiener Relation"	75
7	"periodically correlated" AND "sequences OR processes"	1,740
8	"signal analysis" AND "CS/CS"	3,210
9	"signal processing" AND "CS/CS"	19,000
10	"spatial filtering" AND "CS/CS"	571
11	"spectral redundancy" AND "CS/CS"	1,170
12	computers AND "CS/CS"	17,700
13	cyclic spectrum AND "CS/CS"	9,580
14	cyclostationary OR cyclostationarity	25,000
15	equalization AND "CS/CS"	6,360
16	estimation AND "CS/CS"	20,800
17	filtering AND "CS/CS"	18,400
18	filtering OR smoothing AND "CS/CS"	12,400

19	higher-order OR cumulant AND cyclostationarity	6,040
20	identification AND "CS/CS"	15,700
21	mathematics AND "CS/CS"	10,900
22	prediction AND "CS/CS"	9,540
23	sensing AND "CS/CS"	14,900
24	spectrum AND "CS/CS"	22,200
	TOTAL	237,507

1 "CS/CS" is an abbreviation for "cyclostationary OR cyclostationarity"

Table 3 Nearly Distinct Application Areas[a]

From: Statistically inferred time warping: extending the cyclostationarity paradigm from regular to irregular statistical cyclicity in scientific data

1	communications AND "CS/CS"	21,200
2	modulation AND "CS/CS"	17,000
3	wireless AND "CS/CS"	15,100
4	classification AND "CS/CS"	10,900
5	"cognitive radio" AND "CS/CS"	8,540
6	"radar OR sonar OR lidar" AND "CS/CS"	5,440
7	mechanical AND "CS/CS"	4,770
8	physics AND "CS/CS"	4,539
9	Sun AND "CS/CS"	4,320
10	bearings AND "CS/CS"	3,980

11	vibration OR rotating machines AND "CS/CS"	3,240
12	"aeronautics OR astronautics OR navigation" AND "CS/CS"	3,190
13	"oceanography OR ocean OR maritime OR sea" AND "CS/CS"	3,060
14	medicine AND "CS/CS"	2,990
15	interception AND "CS/CS"	2,270
16	"atmosphere OR weather OR meteorology OR cyclone OR hurricane OR tornado" AND "CS/CS"	2,230
17	economics AND "CS/CS"	2,050
18	autoregression AND "CS/CS"	2,040
19	biology AND "CS/CS"	2,030
20	chemistry AND "CS/CS"	2,020
21	gears AND "CS/CS"	2,000
22	medical imaging OR scanning AND "CS/CS"	1,370
23	"direction finding" AND "CS/CS"	1,170
24	"seismology OR earthquakes OR geophysics OR geology" AND "CS/CS"	1,090
25	walking AND "CS/CS"	990
26	"astronomy OR astrophysics" AND "CS/CS"	864
27	climatology AND "CS/CS"	811
28	"electroencephalography OR cardiography" AND "CS/CS"	742
29	geolocation AND "CS/CS"	676
30	"physiology" AND "CS/CS"	673

31	"rheology OR hydrology" AND "CS/CS"	639
32	plasma AND "CS/CS"	542
33	"cyclic MUSIC"	512
34	"global warming" AND "CS/CS"	369
35	ecology AND "CS/CS"	356
36	galaxies OR stars AND "CS/CS"	313
37	biometrics AND "CS/CS"	309
38	"planets OR moons" AND "CS/CS"	274
39	UAVs AND "CS/CS"	238
40	universe AND "CS/CS"	209
41	cosmology AND "CS/CS"	172
42	"comets OR asteroids" AND "CS/CS"	155
43	"pulsars" AND "CS/CS"	115
44	"SETI OR extraterrestrial" AND "CS/CS"	83
45	quasars AND "CS/CS"	47
	TOTAL	135,628

1 "CS/CS" is an abbreviation for "cyclostationary OR cyclostationarity"

Table 4 Partially Redundant General Subjects[a]

From: Statistically inferred time warping: extending the cyclostationarity paradigm from regular to irregular statistical cyclicity in scientific data

1	cyclostationary OR cyclostationarity	25,000
2	spectrum AND "CS/CS"	22,200
3	estimation AND "CS/CS"	20,800

4	"signal processing" AND "CS/CS"	19,000
5	filtering AND "CS/CS"	18,400
6	computers AND "CS/CS"	17,700
7	identification AND "CS/CS"	15,700
8	sensing AND "CS/CS"	14,900
9	filtering OR smoothing AND "CS/CS"	12,400
10	"exploiting "CS/CS""	11,900
11	mathematics AND "CS/CS"	10,900
12	cyclic spectrum AND "CS/CS"	9,580
13	prediction AND "CS/CS"	9,540
14	"almost "CS/CS""	8,840
15	equalization AND "CS/CS"	6,360
16	higher-order OR cumulant AND cyclostationarity	6,040
17	"signal analysis" AND "CS/CS"	3,210
18	"periodically correlated" AND "sequences OR processes"	1,740
19	"spectral redundancy" AND "CS/CS"	1,170
20	"Cyclic Wiener Filtering" OR "FRESH filtering" OR "frequency-shift filtering"	676
21	"spatial filtering" AND "CS/CS"	571
22	"cyclostationary EOF" OR "cyclostationary empirical orthogonal functions" OR CSEOF	453
23	"almost periodically correlated" AND "sequences OR processes"	352

24	"Gardner relation" OR "Cyclic Wiener Relation"	75
	TOTAL	237,507

1 "CS/CS" is an abbreviation for "cyclostationary OR cyclostationarity"

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